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Abstract: In the context of multidimensional poverty measurement, it seems plausible to assume that when individuals are deprived in some dimensions and non-deprived in the remaining ones, the latter can be allowed to play a non-trivial role in the assessment of those individuals' poverty levels. Yet, this simple and attractive property is violated by virtually all multidimensional poverty indices proposed in the literature so far because they stick to the so-called 'Strong Focus' axiom. This paper characterizes a class of multidimensional poverty indices that allows for certain trade-offs between deprived and non-deprived attributes when measuring individuals' deprivation. The empirical results based on 'Demographic and Health Surveys' from 54 countries suggest that our assessments of multidimensional poverty can differ dramatically when the overly restrictive Strong Focus is abandoned in favor of weaker versions of the axiom.

Keywords: Multidimensional poverty measurement, Axiomatization, Focus axiom, Sensitivity analysis.

JEL Classification: I3; I32; D63; O1

1. Introduction

At the beginning of the 21st century, poverty reduction continues to be one of the greatest challenges faced by policy makers in most parts of the world. Not surprisingly, United Nations' Millennium Development Goal #1 prompts countries to halve the proportion of population living in poverty by the year 2015. Therefore, the targeting of poor individuals and the measurement of poverty levels is a high-priority topic of research with enormous policy implications. In the last years it is becoming increasingly acknowledged that poverty is a multidimensional phenomenon, and many authors have insisted on the necessity of defining multidimensional poverty rather than relying on income or consumption expenditures alone (see, for instance, Bourguignon and Chakravarty 2003:26). This line of research is particularly pertinent at this moment given the fact that international institutions like the European Commission or the United Nations are implementing the multidimensional approach to complement official income poverty measures. Following the definition adopted by the Europe 2020 strategy, Eurostat publishes since 2009 the values of the multidimensional AROPE index (people at-risk-of-poverty rate or social exclusion), and since 2010 the United Nations' Human Development Report publishes the values of the so-called 'Multidimensional Poverty Index' for over a hundred countries all over the world (see Alkire and Santos 2010). These publications have renewed the interest and invigorated the debate on multidimensional poverty measurement (see Ravallion (2011), Alkire, Foster and Santos (2011), Silber (2011)). This paper contributes to this debate.

After the seminal contribution of Sen (1976), the measurement of poverty is commonly divided in two steps: the 'identification step' (*i.e.*: decide who is 'poor' and who is not[1]) and the 'aggregation step' (*i.e.*: summarizing information about 'the poor' into a single

[1]

number). For the sake of clarity, it might be useful to think of the 'aggregation step' as a two-stage procedure. Initially, a 'deprivation assessment' stage determines *how poor* 'poor' individuals are. After that, the 'aggregation step' summarizes individuals' poverty levels into a single number. This subtle but important distinction is motivated by the fact that in a multiattribute framework, the problem of assessing individuals' deprivation levels (*i.e.*: the first stage in the 'aggregation step') is a non-trivial matter which, as will be shown below, has received insufficient attention from the literature and will be the main concern of this paper.

Once the identification step is over, one must typically assess the extent of poverty of those individuals that are deemed 'poor'. In this respect, there are different axioms regulating and mediating the extent to which individuals' poverty levels are affected by their achievements in the different dimensions that are being taken into account. Among these, the so-called 'Focus Axiom' can be considered as one of the cornerstones of poverty measurement. In its single-dimensional version, the axiom precludes the possibility that incomes above the poverty line affect our assessment of the poverty levels in a given population. When it comes to define that axiom in a multidimensional setting there are basically two alternatives with essentially different ethical implications: the Strong and Weak Focus axioms. Assuming

While the identification step is relatively straightforward in the single dimensional case (an income poverty line defines who is poor and who is not), the problem becomes more complicated in a multidimensional framework, and different well-known approaches have been proposed in the literature. Assuming one is able to define dimension-specific poverty thresholds that allow determining whether individuals are deprived or not in the corresponding dimensions, one can define the following identification approaches. According to the 'union approach', an individual should be labelled as 'poor' if s/he is deprived in at least one dimension. At the other extreme, the 'intersection approach' states that an individual is 'poor' if s/he is deprived in all dimensions simultaneously. Since these extreme approaches are likely to over-estimate and sub-estimate respectively the set of individuals that should be considered as 'poor' (particularly when the number of dimensions that are being considered is large), Alkire and Foster (2011) proposed a counting approach based on Atkinson (2003) suggesting that an individual is 'poor' when s/he is deprived in an intermediate number of dimensions that has to be decided by the analyst. Lastly, another identification method is the so-called 'poverty frontier' approach, which basically combines multidimensional distributions to generate a single-dimensional well-being distribution that is later analyzed with traditional income poverty tools.

one is able to define dimension-specific poverty thresholds to determine whether individuals are deprived or not in the corresponding dimensions, the *Strong Focus* axiom demands that poverty measures should be insensitive to *any* change occurring above the different poverty lines. This axiom rules out any possible trade-off between achievements below and above the poverty lines for any given individual. Even if it is a quite stringent requirement that is insensitive to many situations in which the over-achievements in certain dimensions could somehow compensate the low achievements in other dimensions, up to now it has been imposed on virtually all poverty measures proposed in the literature (e.g.: Tsui 2002, Bourguignon and Chakravarty 2003, Chakravarty et al. 2008, Bossert et al. 2013, Alkire and Foster 2011). On the other hand, *Weak Focus* is a less stringent requirement stating that a poverty measure should be insensitive to increases of non-poor individuals' attributes only, therefore leaving room for certain trade-offs between achievements above and below the poverty lines of poor individuals. In this context, Dutta et al. (2003:205) support the reasonableness of that axiom when they state :"*One can argue that there is no reason why, other things remaining the same, a change in the level of over-achievement of an individual in terms of an attribute should not be allowed to affect the assessment of the overall deprivation of that individual*". More generally, the existence of trade-offs between attributes in the assessment of deprivation levels has been a topic of major concern since the times of the 'Basic Needs' approach (see Streeten 1977, Hicks and Streeten 1979) up until the present day (see, among many others, Dowrick et al. 2003 or Ravallion 2012).

The fact that virtually all multidimensional poverty measures proposed in the literature so far satisfy the Strong Focus axiom is truly remarkable, since there are many circumstances in which one might want to allow over-achievements to exert some kind of influence when assessing individuals' overall deprivation levels. Consider a stylized setting in which poverty

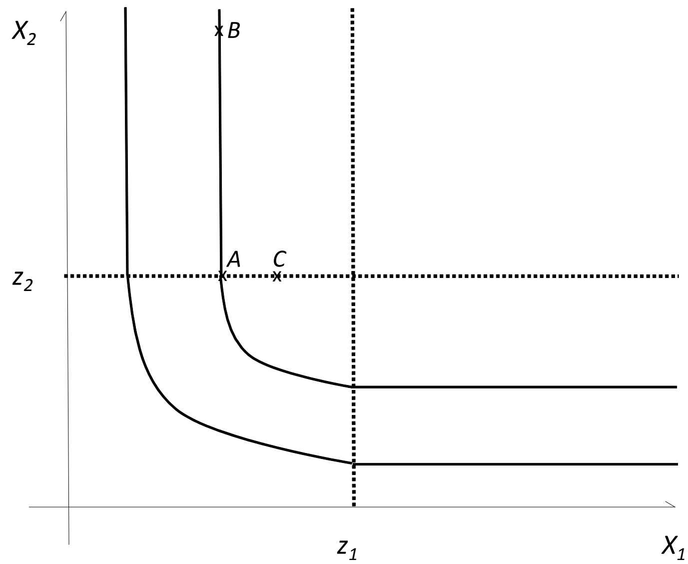

Figure 1: Iso-poverty contours under the Strong Focus axiom for the case $k = 2$.

levels are assessed through the dimensions of health and income. Using the union approach, an individual A with poor health and an income on the corresponding poverty line (typically a very low income) can be reasonably considered as a 'poor' person. Another individual B with the same poor health but with a moderate income level could also be considered to be poor according to the union approach *but not as poor as* individual A, because her higher income level might allow her to somehow compensate for her poor health status and enjoy a better standard of living. Interestingly, since virtually all multidimensional poverty indices that have been proposed in the literature so far are insensitive to this kind of interaction between dimensions, they would –somewhat surprisingly– consider individual A to be *exactly as poor as* individual B (see Figure 1)[2] .

The main goal of this paper is to propose a new conceptualization of multidimensional poverty in such a way that achievements above the poverty line are allowed to play a non-

[2] The only exception to that rule we are aware of is that of Lugo and Maasoumi (2008). However, the indices proposed in that paper have an important limitation that will be explained in the discussion section.

trivial role in the assessment of individuals' deprivation levels. More specifically, we will generalize the existing multidimensional poverty measures in such a way that they will be allowed to violate the Strong Focus axiom while satisfying its Weak version. In terms of the stylized setting presented in Figure 1, we want to extend the family of multidimensional poverty measures in such a way that the poverty level of individual B can be considered to be the same as the poverty level of an individual C which is situated somewhere to the right of individual A, or –if deemed appropriate– as the poverty level of individual A itself.

While there are many compelling reasons to allow over-achievements playing a non-trivial role in the assessment of individuals' deprivation levels, it is not straightforward to specify the extent to which a certain attribute should be traded-off by another one. Among other things, this would require determining empirically the extent to which these attributes are complements or substitutes, an issue for which there does not seem to be a standard procedure (Alkire and Foster 2011:486). In other words: there are widely varying degrees in which the Strong Focus axiom can be relaxed in favor of its weak version. In face of such a daunting task, a decision maker might be uncertain and could prefer to introduce a certain degree of underspecification: rather than arbitrarily fixing the values of parameters governing trade-offs between deprived and non-deprived attributes, she might prefer to let them freely move within certain regions of those parameters' space denoted as 'admissible sets' (call them Λ)[3] . This paper introduces different tools to assess the extent to which the set of admissible rankings derived from the choice of Λ differs with respect to the 'status quo' ranking derived from the Strong Focus axiom – so the reliability and robustness of the later can be fully explored. In particular, we will investigate the pace

[3] Similar parameter underspecification techniques to explore the robustness of results have also been recently used in the literature of well-being measurement (*e.g.*: Saisana et al. 2005, Cherchye et al 2008, Permanyer 2011a,b, Foster, McGillivray and Seth 2013).

at which the dissimilarity between the rankings associated to the Strong and Weak Focus assumptions increases as we gradually enlarge the size of the admissible sets (i.e.: as we increasingly weaken the Strong Focus axiom). Interestingly, the methodology presented in this paper allows modelling different degrees of complementarity / substitutability for alternative couples of attributes – an improvement with respect to the current state of the literature, that requires attributes to be all complements or all substitutes with a strength that is uniform across all pairs.

Using the same 54 Demographic and Health Surveys that were used in the calculation of UNDP's Multidimensional Poverty Index, we show an application of our methodology to test the robustness of 'Strong Focus poverty rankings' to alternative implementations of the Weak Focus axiom. *Inter alia*, the results shown in this paper suggest that our assessments of multidimensional poverty levels can differ dramatically when making some room for trade-offs between deprived and non-deprived attributes.

The paper is structured as follows. Section 2 will present some basic definitions and notation that will be used throughout the paper. Our methodology and its axiomatic characterization will be presented in section 3. Section 4 presents some sensitivity analysis tools that will be used in the empirical application. In section 5 we turn to our empirical application and in section 6 we present some substantive comments on the implications of our results. The proofs are relegated to the Appendix.

2. Preliminary notation and definitions

We introduce some basic definitions that will be used throughout the paper. $\mathbb{R}^q, \mathbb{R}^q_+, \mathbb{R}^q_{++}$ are the $q-$dimensional Euclidean space and its nonnegative and positive counterparts respectively. We consider k well-being dimensions (which might be referred to as 'attributes' or, simply, 'dimensions', and which are labelled as $\{1, \ldots, k\} =: K$) and n individuals. The achievement of individual i in attribute j will be denoted by x_{ij}. From now on, we impose that $x_{ij} \geq 0$, an almost universal assumption in both unidimensional and multidimensional poverty measurement[4] . The vector $\mathbf{x}_i = (x_{i1}, \ldots, x_{ik}) \in \mathbb{R}^k_+$ will be called *achievement vector* for individual i. Given the fact that we are considering one achievement vector for each individual, we will define an *achievement matrix* as a $n \times k$ matrix with non-negative entries.

For each attribute j we consider a poverty threshold $z_j > 0$ representing a minimum quantity of that attribute that is needed for subsistence – which in this paper we will consider as exogenously given. In this context, we say that individual i is *deprived* in attribute j (or that j is a *meagre* attribute for individual i) whenever $x_{ij} \leq z_j$. We will denote by $z = (z_1, \ldots, z_k) \in \mathbb{R}^k_{++}$ the vector of poverty thresholds. Hence, we say that a given achievement vector $\mathbf{x}_i$ does belong to a $p-$dimensional poverty space (with $p \in \mathbb{N}$, $0 \leq p \leq k$) if the number of attributes falling below the poverty thresholds is exactly p. For instance, in the case of two attributes, one can find 0, 1 or 2-dimensional poverty spaces, indicating that an individual can be deprived in none, one or two dimensions respectively.

Whenever an individual is deprived in a given attribute there are several ways of capturing the extent of that deprivation, usually referred to as 'deprivation shortfall' or 'deprivation gap'. The different multidimensional poverty indices introduced in the literature so far use alternative functional forms to capture those gaps. These are shown in Table 1 together with the corresponding multidimensional poverty indices that are derived from them.

4 Tsui (2002:82) concludes that multidimensional poverty measurement is severely limited when some of the arguments can take negative values.

Paper	Notation	Formula	Range	MD Poverty Index
Tsui (2002)	g_{ij}^{T1}	$z_j/Min\{x_{ij}, z_j\}$	$R_{T1}=[1, +\infty)$	$P^{T1} = \frac{1}{n}\sum_{i=1}^{n}\left[\prod_{j=1}^{k}\left(g_{ij}^{T1}\right)^{\alpha_j} - 1\right]$
Tsui (2002)	g_{ij}^{T2}	$\ln(z_j/Min\{x_{ij}, z_j\})$	$R_{T2}=[0, +\infty)$	$P^{T2} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{k}\delta_j g_{ij}^{T2}$
Tsui (2002)	g_{ij}^{T3}	$z_j - Min\{x_{ij}, z_j\}$	$R_{T3}=[0, +\infty)$	$P^{T3} = \frac{1}{n}\sum_{i=1}^{n}\left(\prod_{j=1}^{k} e^{r_j g_{ij}^{T3}} - 1\right)$
Tsui (2002)	g_{ij}^{T4}	$z_j - Min\{x_{ij}, z_j\}$	$R_{T4}=[0, +\infty)$	$P^{T4} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{k} c_j g_{ij}^{T4}$
B&C (2003)	g_{ij}^{BC}	$Max\left\{\frac{z_j - x_{ij}}{z_j}, 0\right\}$	$R_{BC}=[0, 1]$	$P^{BC} = \frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=1}^{k} w_j\left(g_{ij}^{BC}\right)^{\theta}\right]^{\beta/\theta}$
C&D&S (2008)	g_{ij}^{W}	$\ln(z_j/Min\{x_{ij}, z_j\})$	$R_{W}=[0, +\infty)$	$P^{W} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{k} g_{ij}^{W}$
A&F (2011)	g_{ij}^{AF}	$Max\left\{\frac{z_j - x_{ij}}{z_j}, 0\right\}$	$R_{AF}=[0, 1]$	$P^{AF} = \frac{1}{nk}\sum_{i\in P}\sum_{j=1}^{k}\left(g_{ij}^{AF}\right)^{\alpha}$

Table 1. Deprivation shortfalls with their functional forms, ranges and corresponding multidimensional poverty measures. B&C stands for Bourguignon and Chakravarty; C&D&S for Chakravarty, Deutsch and Silber and A&F for Alkire and Foster. The parameters appearing in the table must satisfy the following restrictions (see the original papers for details): $\alpha_j \geq 0 \forall j$ and α_j chosen so that the function $\Pi_j y_j^{-\alpha_j}, y_j \in (0, 1]$ is convex with respect to $(y_1, \ldots, y_k)$; $\delta_j \geq 0 \forall j$; $r_j \geq 0 \forall j$ with at least one r_j strictly positive and r_j chosen so that $\Pi_j e^{r_j(z_j - x_{ij})}$ is convex; $c_j \geq 0 \forall j$ with at least one c_j strictly positive; $w_j \geq 0 \forall j$, $\theta, \beta > 0$;$\alpha \geq 0$. In the Alkire and Foster index, P stands for the set of individuals labeled as 'poor' according to the number of dimensions in which they are deprived.

As shown in Table 1, all deprivation gaps are defined as

$$g_{ij} = f(x_{ij}, z_j) \tag{1}$$

for some function $f : \mathbb{R}_+ \times \mathbb{R}_{++} \to \mathbb{R}$. Recall that the range of values of the deprivation gaps g_{ij} and the corresponding multidimensional poverty measures are quite different. From now onwards, when we refer to any of those possible ranges we will simply write R (that is: $R \in \{[1, +\infty), [0, +\infty), [0, 1]\}$). Once the functional form of the deprivation gap $f(.,.)$ has been chosen, we define

$$g_{\min} := \min_{x_{ij} \geq 0} f(x_{ij}, z_j) \tag{2}$$

as the value of the smallest possible deprivation gap (which, by construction, is the same for any dimension $j \in \{1, \ldots, k\}$). In other words, $g_{\min}$ is the smallest possible value in the range of admissible values for the function f. As can be seen in Table 1 we have that $g_{\min} = 0$ for all poverty measures presented in the literature except for the first index presented by Tsui (2002), where $g^{T1}_{\min} = 1$.

We will denote by G a generic $n \times k$ matrix with the values of the deprivation gaps $g_{ij} \in R$ – referred to as *deprivation matrix*. The set of $n \times k$ deprivation matrices will be denoted as $\mathcal{G}^{n \times k}$ and we will write $\mathcal{G} = \bigcup_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \mathcal{G}^{n \times k}$. As can be seen in Table 1, all existing multidimensional poverty indices are defined as functions $P : \mathcal{G} \to \mathbb{R}$ that for a given deprivation matrix determine the level of poverty in that distribution. With this notation, the indices in Table 1 can be written as $P = P(G)$, so they are not making room for an eventual role of over-achievements in the assessment of poverty levels. A consequence of imposing the Strong Focus axiom is that the iso-poverty contours in $p-$dimensional poverty spaces ($p < k$) will be parallel to the $k - p$ axes where the corresponding attributes are non-meagre. Figure 1 – which is an adaptation of Figure 3 in Bourguignon and Chakravarty (2003) – illustrates this fact for the case where $k = 2$: under Strong Focus, individuals A and B are located in the same iso-poverty contour even if both are equally deprived in dimension 1 and individual B is an over-achiever in dimension 2.

Since we aim to explore the role that achievements above the poverty line can have on our multidimensional poverty assessments, we need to introduce some assumptions and definitions. Our initial assumption is that there is an upper bound for each attribute, that is: there is a maximum achievement level for each attribute. In formal terms, we are assuming that for each attribute j there exists a constant U_j such that $x_{ij} \leq U_j < \infty$ for all x_{ij}. This assumption is very reasonable for most well-being dimensions that are typically incorporated in multidimensional poverty assessments (see section 5). In most cases, the task of finding a reasonable bound U_j above which further increases do not make a sensible difference in poverty assessments does not seem to be unsurmountable. For instance, the censoring of distributions is a common practice in several analysis (*e.g.*: many multidimensional indices censor some of their components in order to avoid the distortion that the inclusion of extreme values would entail on the corresponding normalized distributions – this is the case of the well-known Human Development Index, the Environmental Sustainability Index or the World Economic Forum Gender Gap Index, to mention a few).

For any $x_{ij} \geq 0$ and any $z_j > 0$ we define the corresponding 'excess gap' as

$$e_{ij} := \left\{ \begin{array}{c} \frac{x_{ij}-z_j}{U_j-z_j} \text{ if } x_{ij} \geq z_j \\ 0 \text{ otherwise} \end{array} \right\}$$

Recall that e_{ij} compares the observed excess from the poverty line with respect the maximal over-achievement that is feasible under the domain constraints[5] . By definition $e_{ij} \in [0,1]$; when $x_{ij} = z_j$, $e_{ij} = 0$ (there is no over-achievement when we are at the poverty line) and when $x_{ij} = U_j$, $e_{ij} = 1$ (over-achievement is maximal when x_{ij} reaches the upper bound). Figure 2 illustrates graphically the values of the deprivation gaps $g_{ij}^{BC} = g_{ij}^{AF} \in [0,1]$ used in Bourguignon and Chakravarty (2003) and Alkire and Foster (2011) and the excess

[5] If it is deemed more appropriate, the excess gaps can also be defined in non-linear ways. Such changes would not alter the characterization results presented in this paper.

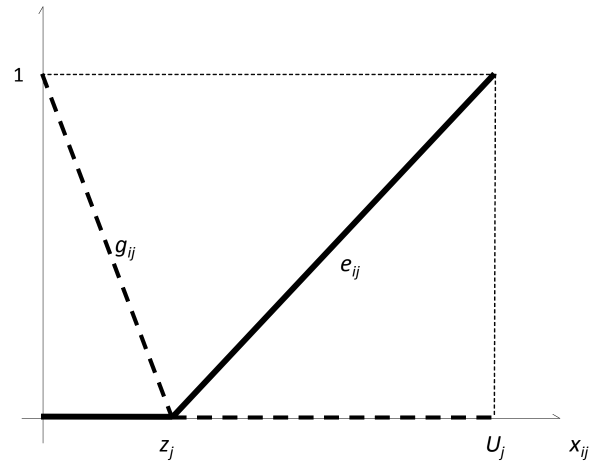

Figure 2: Graph of the poverty shortfalls g_{ij}^{BC} and the excess gaps e_{ij} for the different values of x_{ij}.

gaps e_{ij} for the different values of x_{ij}. For any individual (say, i), we define the corresponding deprivation and excess vectors as $\mathbf{g}_i := (g_{i1}, \ldots, g_{ik})$ and $\mathbf{e}_i := (e_{i1}, \ldots, e_{ik})$ respectively.

The arguments of the multidimensional poverty measures proposed in this paper will consist of ordered pairs of $n \times k$ matrices (G, E), where the elements of the first and second matrices are deprivation and excess gaps respectively and where the following restriction holds: whenever $g_{ij} > g_{\min}$, then $e_{ij} = 0$ (that is: if individual i is deprived in attribute j, then the corresponding excess gap must equal zero). If we denote the set of such ordered pairs of matrices as $(\mathcal{G} \times \mathcal{E})^{n \times k}$, we can write $\mathcal{G} \times \mathcal{E} = \bigcup_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} (\mathcal{G} \times \mathcal{E})^{n \times k}$.

Definition 1. A *multidimensional poverty index* P is a non-trivial function $P : \mathcal{G} \times \mathcal{E} \to \mathbb{R}$.

This definition is extremely general and undemanding and it includes all multidimensional poverty indices presented in the literature (shown in Table 1) as particular cases. In the next section we are going to impose some "reasonable" restrictions (*i.e.*: axioms) on P so as to pin down an explicit functional form that can be useful for empirical analysis.

To conclude this section, we introduce some notation that will be used to present the axioms in a clear cut way. The set of $n \times k$ deprivation and excess matrices whose rows are the same will be denoted as $\mathcal{G}_S^{n\times k}$ and $\mathcal{E}_S^{n\times k}$ respectively. Let $K = \{1, \dots, k\}$ be the set of attributes / dimensions we are taking into account. Then we will denote by G^H the submatrix of $G \in \mathcal{G}_S^{n\times k}$ containing the individuals' deprivations for the attributes in a non-empty $H \subseteq K$, and by G^{H^C} the submatrix of $G \in \mathcal{G}_S^{n\times k}$ containing the remaining attributes. For any $n \times k$ matrix with real entries M, any attribute $j \in K$ and any constant $a \in \mathbb{R}$, we will denote by (M^{-j}, a) the $n \times k$ matrix that is exactly as M except for column j, which has been substituted by a column where all its components are equal to a.

3. Characterization results

The first axiom we introduce in this paper – *Subgroup Decomposability* – restricts the class of multidimensional poverty indices we are going to consider. That axiom states that if a population is partitioned into several subgroups with respect to a certain characteristic (e.g.: sex, age, place of residence, etc.) then overall poverty is the population weighted average of the subgroup poverty levels. In formal terms

Subgroup Decomposability. Let $(G_1, E_1) \in \mathcal{G}^{n_1\times k}, \dots, (G_p, E_p) \in \mathcal{G}^{n_p\times k}$ be a list of deprivation and excess matrices for p disjoint subpopulations. Then, poverty on the overall population can be written as $P(G_1, \dots, G_p, E_1, \dots, E_p) = \sum_l \frac{n_l}{n} P(G_l, E_l)$, where $n = \sum_l n_l$.

This property allows identifying subgroups where poverty is particularly high and evaluating their contribution to overall poverty levels. Therefore, it is an extremely useful property in applied analysis, where policy-makers need to target and monitor the most vulnerable groups. In fact, *Subgroup Decomposability* is such an intuitive and useful property that it has been imposed on *all* multidimensional poverty indices presented in the literature so far (see Table 1). A trivial implication of this axiom is that a poverty index P can be written as

$$P = \frac{1}{n}\sum_{i=1}^{n} p(\mathbf{g}_i, \mathbf{e}_i)$$

where $p(\mathbf{g}_i, \mathbf{e}_i)$ can be interpreted as the level of multidimensional poverty associated with individual i given the achievement and poverty thresholds vectors $\mathbf{x}_i$ and z.

Continuity. P is a continuous function in its arguments.

This is an almost universal assumption satisfied by most poverty measures and, broadly speaking, most socio-economic indices. It requires that small changes in the achievements of individuals produce small changes in the corresponding deprivations. Stated otherwise: the deprivation level does not abruptly change as individuals' achievements are slightly altered. Among other things, this property ensures that deprivation levels will not be dramatically affected by measurement errors. Despite its appeal and intuitiveness, there are at least two well-known poverty indices that do not satisfy *Continuity.* One of them is the Headcount Ratio[6] and the other one is the multidimensional poverty index M_α proposed by Alkire and Foster (2011) which was used in the definition of UNDP's Multidimensional Poverty Index (MPI)[7] .

[6] In the income poverty framework, the headcount ratio is the proportion of individuals below the poverty line (i.e.: $H := q/n$, where q is the number of poor individuals). As pointed out by Sen (1976), H is a discontinuous index: slight changes in incomes below the poverty line can lead to sudden jumps in the poverty measure.

[7] Alkire and Foster (2011) identify poor individuals using the so-called dual cutoff approach. One form of cutoff is used within each dimension to identify whether a person is deprived in that dimension, and a second cutoff across dimensions is used to identify the poor by counting the dimensions in which a person is deprived. However this introduces a discontinuity in the measure around the cutoff points that might go against our intuitions in certain cases. One of the consequences of this discontinuity is that certain regressive Pigou-Dalton transfers might decrease poverty rather than increasing it, a somewhat disturbing result. Consider the following illustrative example in a two-member society with $k = 4, q = 3$ (i.e.: an individual must be deprived in at least three attributes to be considered as poor), $z_i = 0.3$ for all $i \in \{1, 2, 3, 4\}$ and $\mathbf{x}_1 = (0.1, 0.1, 0.11, 0.1), \mathbf{x}_2 = (0.1, 0.1, 0.29, 1)$. Suppose there is a regressive Pigou-Dalton transfer where individual 1 transfers 0.01 units of the third attribute to individual 2. After the transfer one has that $\mathbf{x}'_1 = (0.1, 0.1, 0.1, 0.1), \mathbf{x}'_2 = (0.1, 0.1, 0.3, 1)$, so the second individual is no longer poor. According to the Alkire-Foster class of poverty measures M_α, poverty has decreased after the transfer.

Homotheticity. For any $(G_1, E), (G_2, E) \in (\mathcal{G} \times \mathcal{E})^{n \times k}$ and any $\lambda \in R$ one has that $P(G_1, E) \geq P(G_2, E) \Leftrightarrow P(\lambda G_1, E) \geq P(\lambda G_2, E)$, where $\lambda G_1, \lambda G_2$ are the $\lambda-$scalings of deprivation matrices G_1, G_2.

Consider two moments in time: t_1 and t_2. In t_1 we compare the multidimensional poverty levels of two equal-sized populations that have the same matrix of excess gaps. In t_2, the deprivation gaps are scaled up or down by the same amount, all else remaining the same. *Homotheticity* states that under such change, the ranking of these two societies in terms of multidimensional poverty should remain the same. Stated otherwise: a multidimensional poverty ordering must be preserved when the corresponding deprivation gaps are changed proportionally. Once again, this is a highly plausible axiom that is satisfied by all multidimensional poverty indices introduced in the literature (see Table 1).

Weak Dimension Separability. For all sets of attributes $H \subseteq K$ and for all $F, G, \widehat{F}, \widehat{G} \in \mathcal{G}_S^{n \times k}$, $E \in \mathcal{E}_S^{n \times k}$ such that $F^H = G^H, \widehat{F}^H = \widehat{G}^H, F^{H^C} = \widehat{F}^{H^C}$ and $G^{H^C} = \widehat{G}^{H^C}, P(F, E) \geq P(G, E) \Leftrightarrow P(\widehat{F}, E) \geq P(\widehat{G}, E)$.

Consider a hypothetical scenario in which we compare equal-sized populations where all individuals are identical in the sense that they all have the same deprivation and excess vectors. Imagine that after a period of time, the deprivations felt by individuals with respect to some attributes do not change, while the rest of the deprivations vary (with the excess vectors remaining the same). In this scenario, it is reasonable to expect that the levels of multidimensional deprivation should depend on the variation of the attributes that have changed, but not on the others that have remained constant. This is what *Weak Dimension Separability* imposes, demanding that the ranking of the two populations in times t_1 and t_2 is independent of the level of deprivation in the dimensions that do not change over time.

Monotonicity on Deprivation Gaps. Let $(F, E), (G, E) \in (\mathcal{G} \times \mathcal{E})^{n \times k}$. If $F \neq G$ and $f_{ij} \geq g_{ij}$ for all $i \in \{1, \ldots, n\}, j \in \{1, \ldots, k\}$, then $P(F, E) > P(G, E)$.

This assumption simply states that a multidimensional poverty index should be increasing in its deprivation arguments, that is: if the deprivation felt by any individual in any attribute increases and all else remains the same, overall deprivation should increase as well.

Independence. Consider any $(F, U), (G, V) \in (\mathcal{G} \times \mathcal{E})^{n \times k}$, any attribute $j \in \{1, \ldots, k\}$ and any $a \in [0, 1]$. If $P((F^{-j}, g_{\min}), (U^{-j}, 0)) = P((G^{-j}, g_{\min}), (V^{-j}, 0))$, then $P((F^{-j}, g_{\min}), (U^{-j}, a)) = P((G^{-j}, g_{\min}), (V^{-j}, a))$.

This property demands the following. Assume we are comparing two equal-sized populations in time t_1 where all individuals' achivement level in attribute j equals the corresponding poverty line z_j and assume that the overall deprivation for both populations is exactly the same. Imagine that after a period of time, all individuals' achievement level in attribute j is increased by the same amount above the poverty line and all else remains the same. Then it is reasonable to expect that the overall deprivation for both populations will continue to be the same (even if it has changed with respect to its original level in time t_1). In other words, *Independence* requires that when equals are added to equals, the results that are obtained should also be equal.

Theorem 1. If a multidimensional poverty index P satisfies *Subgroup Decomposability, Continuity, Homotheticity, Weak Dimension Separability, Monotonicity on Deprivation Gaps* and *Independence*, then it can be written as

$$P(G, E) = \frac{1}{n} \sum_{i=1}^{n} \psi \left(\left[\sum_{j=1}^{k} \left(g_{ij} \prod_{l=1}^{k} \varphi_{jl}(e_{il}) \right)^{\theta} \right]^{1/\theta} \right) \tag{3}$$

where ψ is a continuous increasing function, $\varphi_{jl}(.)$ are some continuous functions and $\theta > 0$.

Proof: See the Appendix.

Remark 1. The multidimensional poverty index shown in equation (3) is a generalization of *all* multidimensional poverty indices presented in Table 1 (*i.e.*: choosing the appropriate functions ψ, φ_{jl} and parameter θ in equation (3), we can obtain all indices shown in Table 1)[8] . For the sake of simplicity, in the rest of the paper we will make the following assumption.

Consistency Condition. When imposing the Strong Focus axiom to the poverty index $P(G, E)$ shown in equation (3), we obtain the poverty indices shown in Table 1.

The sole purpose of this assumption is to restrict our attention to generalized versions of the multidimensional poverty indices presented in Table 1. Even if it is an *ad hoc* restriction that precludes the existence of other functional forms not encapsulated in Table 1, it is sufficient for the main purpose of this paper, to wit, explore the possibility of allowing over-achievements to play a non-trivial role in the assessment of individuals' multidimensional poverty levels. In future research, it will be interesting to relax this assumption to enrich even further the class of multidimensional poverty indices satisfying the Weak Focus axiom.

Remark 2. As is shown in Theorem 1, in our extended framework each individuals' deprivation gap for a given attribute (g_{ij}) is 'corrected' (*i.e.*: modified) by a certain factor that depends on the extent to which that individual is non-deprived on the other attributes. The extent of that correction is mediated by the continuous functions $\varphi_{jl} : [0, 1] \to \mathbb{R}$, which

[8] P^{T2}, P^{T4}, P^{W} are obtained from equation (3) when $\psi(x) = x, \theta = 1$. P^{T3} is obtained when $\psi(x) = e^x - 1, \theta = 1$. P^{T1} is obtained as a limiting case from equation (3) when $\psi(x) = x - 1, \theta \to 0$. P^{BC} is obtained when $\psi(x) = x^\beta$. Finally, P^{AF} is obtained when $\psi(x) = x^\beta$ and $\theta = \beta$. In all cases, it is needed that the functions $\varphi_{jl}(x) = c_{jl}$ for certain constants c_{jl}.

from now onwards will be referred to as *correction functions*. The values of $(1-\varphi_{jl}(e_{il}))100$ could be interpreted as the percent reduction of the deprivation gap g_{ij} when we take into account the excess gap in attribute l (e_{il}). Under Strong Focus one would have that $\varphi_{jl}(e_{il}) = 1$, so $(1-\varphi_{jl}(e_{il}))100 = 0$, and there is a 0% reduction of the corresponding deprivation gap g_{ij}.

Imposing some restrictions, the functional form of the multidimensional poverty index has been narrowed down considerably. However, more axioms are necessary to pin down an explicit functional form for the correction functions $\{\varphi_{jl}\}$ so that empirical applications are possible.

Monotonicity on Excess Gaps. Let $(G,U),(G,V) \in (\mathcal{G}\times\mathcal{E})^{n\times k}$. If $U \neq V$ and $u_{ij} \geq v_{ij}$ for all $i \in \{1,\ldots,n\}, j \in \{1,\ldots,k\}$, then $P(G,U) \leq P(G,V)$.

This property ensures that when the achievement level in a non-meagre attribute is increased even further, then the corresponding overall deprivation level does not increase. This is the basic intuition that motivates this paper: increases in non-meagre attributes are allowed to play a non-trivial role in the assessment of individuals' deprivation levels.

Uniform Scale Invariance. Let $q \in \{1,\ldots,k\} = K$ and let $\boldsymbol{\epsilon}_i = \{(\epsilon_{il})\}_{l\in K}, \boldsymbol{\varepsilon}_i = \{(\varepsilon_{il})\}_{l\in K}$ be two excess vectors such that $\epsilon_{il} = \varepsilon_{il}$ for all $l \in K\backslash\{q\}$. Define $\boldsymbol{\epsilon}_i^t = \{(\epsilon_{il}^t)\}_{l\in K}$ such that $\epsilon_{il}^t = \epsilon_{il}$ for all $l \in K\backslash\{q\}$; $\epsilon_{iq}^t = t\epsilon_{iq}$ and define $\boldsymbol{\varepsilon}_i^t = \{(\varepsilon_{il}^t)\}_{l\in K}$ such that $\varepsilon_{il}^t = \varepsilon_{il}$ for all $l \in K\backslash\{q\}$; $\varepsilon_{iq}^t = t\varepsilon_{iq}$, with $t > 0$ in such a way that $t\epsilon_{iq}, t\varepsilon_{iq} \in [0,1]$. If one defines $U, V, U^t, V^t \in \mathcal{E}_S^{n\times k}$ as the excess matrices where all rows are the same corresponding to the excess vectors $\boldsymbol{\epsilon}_i, \boldsymbol{\varepsilon}_i, \boldsymbol{\epsilon}_i^t$ and $\boldsymbol{\varepsilon}_i^t$ respectively, then $\xi(P(G,U^t)) - \xi(P(G,V^t)) = h(\xi(P(G,U)) - \xi(P(G,V)), t)$ for any $G \in \mathcal{G}_S^{n\times k}$, where $h: \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ is a function and $\xi: \mathbb{R}\to\mathbb{R}$ is an increasing function.

This is also a standard property used in utility or poverty measurement (see, for instance, Eichhorn and Gleissner (1988:24), or Chakraborty, Pattanaik and Xu (2008)). Suppose that we compare the individual poverty levels between two poor persons i_1 and i_2 where the achievement vectors $\mathbf{x}_{i_1}, \mathbf{x}_{i_2}$ are exactly the same except for their value in dimension q, where both individuals are non-poor ($x_{i_1q}, x_{i_2q} > z_q$). Imagine that there is an equiproportionate increase in the excess gaps in dimension q for both individuals. Then, up to a monotonic transformation, the difference between the individual poverty levels in the new situation will change by an amount that will depend exclusively on the initial difference in poverty levels and the proportionality factor by which the achievement level in dimension q was increased.

Theorem 2. Under the *Consistency condition*, a multidimensional poverty index P as shown in equation (3) satisfies *Monotonicity on Excess Gaps* and *Uniform Scale Invariance* if and only if the corresponding correction functions φ_{jl} can be written as

$$\varphi_{jl}(e_{il}) = 1 + (\lambda_{jl} - 1)e_{il}^{\gamma} \tag{4}$$

where $\lambda_{jl} \in (0, 1]$ and $\gamma > 0$.

Proof: See the Appendix.

Remark 3. Applying Theorems 1 and 2, we obtain generalized versions of all multidimensional poverty measures introduced in the literature (see Table 1): it suffices to 'correct' the deprivation gaps g_{ij} multiplying by the functions $\varphi_{jl}(.)$ shown in equation (4). These new indices are allowed to violate the Strong Focus axiom and satisfy its weaker version. Therefore, we are not proposing a specific multidimensional poverty index but a generalization of *all* the indices presented in Table 1.

Remark 4. Parameter γ measures the lack of sensitivity of the multidimensional poverty index to achievements above the poverty line. The larger the value of γ, the smaller the sensitivity to such over-achievements. In the limit, as $\gamma \to \infty$, the poverty index $P(G, E)$ satisfies the Strong Focus axiom.

Remark 5. Recall that, since $\varphi_{jl}(1) = \lambda_{jl}$, the values of $(1 - \lambda_{jl})100$ represent the maximal percentual reduction of the deprivation gap g_{ij} that is possible when the excess gap in attribute l reaches its maximum level ($e_{il} = 1$). By definition, the smaller the values of the λ_{jl}, the larger the extent to which deprivations in dimension j can be compensated by over-achievements in dimension l. Therefore, the values of the λ_{jl} are highly related to the degree of complementarity / substitutability between attributes j and l.

Remark 6. Given the fact that larger values of k might lead to problems of over-parametrization[9] , we propose different simplification strategies. The first one is stated as follows: since the $\{\lambda_{jl}\}$ can be thought of as proxies of the degree of complementarity / substitutability between attributes j and l (*i.e.*: the lambdas are defined for *couples* of attributes), then it is reasonable to assume that $\lambda_{jl} = \lambda_{lj}$ for all $j, l \in \{1, \ldots, k\}$. This halves the number of parameters that should be taken into account. The second approach avoids taking a particular stance in the difficult choice of the $\{\lambda_{jl}\}$ by allowing for parameter underspecification, an issue to which we now turn.

4. Sensitivity Analysis

The multidimensional poverty indices introduced in this paper require specifying the values of the parameters $\{\lambda_{jl}\}$ that measure the extent to which an achievement level in a given dimension above the corresponding poverty threshold can compensate and lower individual's deprivation gaps in other dimensions. Since these parameters can be thought of as proxies

[9] In multidimensional poverty analysis, each additional dimension typically involves the difficult choice of several parameters. Moreover, for a given k there are $k(k-1)$ lambdas $\{\lambda_{jl}\}$ that must be chosen.

of the degree of complementarity / substitutability between couples of attributes, in case of strong complementarity the corresponding λ_{jl} should be close to 1, and large substitutability levels should lead to values close to 0. Unfortunately, there does not seem to be a standard procedure for determining the extent of complementarity and substitutability across poverty dimensions (Alkire and Foster 2011: 486). Therefore, it is not entirely clear how these values should be chosen in an empirical exercise: while the choice of $\lambda_{jl} = 1$ for all $j \neq l$ seems to be unduly restrictive, there are little clues on how to determine their specific value. In face of such a daunting task, a decision-maker might be uncertain and prefer to introduce some degree of parameter underspecification.

4.1 The two country case

Imagine we want to compare the poverty levels in two countries A and B taking into account $k = 3$ dimensions[10] (as will be the case in the empirical section). Given the fact that the values of $P(G_A, E_A), P(G_B, E_B)$ will be partly determined by the specific choice of the parameters λ_{jl} for $j, l \in \{1, 2, 3\}, j \neq l$, the least compromising alternative is to consider what happens when *all* possible values of the λ_{jl} are taken into account. As suggested in Remark 6, in order to simplify our analysis we will assume that $\lambda_{jl} = \lambda_{lj}$ for all $j, l \in \{1, 2, 3\}, j \neq l$, so in the rest of the paper we will be dealing with three parameters only: $\lambda_{12}, \lambda_{13}$ and λ_{23}. In that case, one can construct a diagram like the one shown in Figure 3. The surface inside the unit cube $[0, 1]^3$ represents the set of triples $(\lambda_{12}, \lambda_{13}, \lambda_{23})$ for which $P(G_A, E_A) = P(G_B, E_B)$. The set of points inside the unit cube on one side of the surface represent the set of triples $(\lambda_{12}, \lambda_{13}, \lambda_{23})$ for which $P(G_A, E_A) > P(G_B, E_B)$ and the points on the other side of the surface yield the opposite ranking. Recall that the vertex $(\lambda_{12}, \lambda_{13}, \lambda_{23}) = (1, 1, 1)$ represents

[10] The ideas introduced in this paper can be applied as well for any $k > 3$. For the sake of simplicity, however, we will focus on the case $k = 3$.

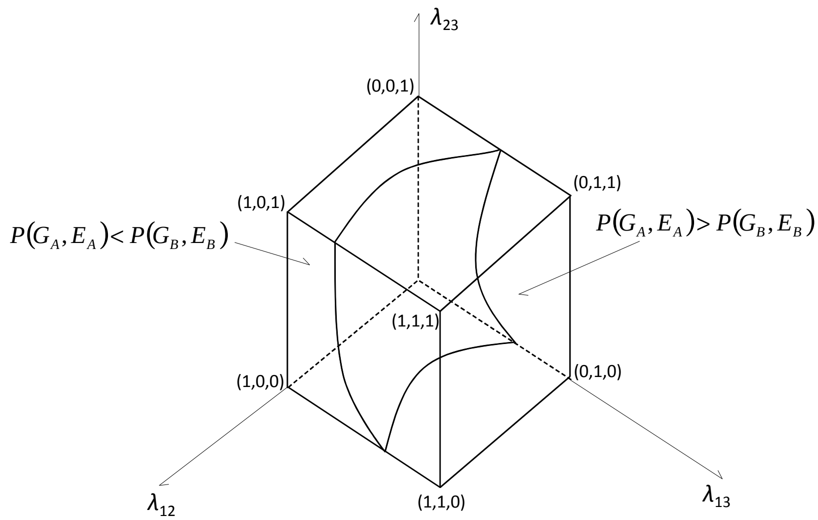

Figure 3: Comparing $P(G_A, E_A), P(G_B, E_B)$ in the $(\lambda_{12}, \lambda_{13}, \lambda_{23})$-space.

the Strong Focus axiom in which no trade-offs between meagre and non-meagre attributes are allowed. On the other hand, the vertex $(\lambda_{12}, \lambda_{13}, \lambda_{23}) = (0, 0, 0)$ represents the opposite extreme case in which *any* poverty shortfall in one dimension can be eventually compensated by a sufficiently large excess in the other dimension. By means of this kind of diagram it is possible to have a more complete picture of the extent to which multidimensional poverty is larger in one country than another and to understand the role played by the trade-offs between meagre and non-meagre attributes.

4.2 The multiple country case

When the previous analysis has to be extended for the case of many countries (say, M), matters can become more complicated because diagrams like the one shown in Figure (3) can be particularly difficult to read (there might be as many as $M(M-1)/2$ surfaces to plot in the unit cube). In order to overcome this problem, in this paper we suggest the following alternative approach: for any $\boldsymbol{\lambda} := (\lambda_{12}, \lambda_{13}, \lambda_{23}) \in [0,1]^3$, we compare the $M-$country ranking that is obtained from that specific choice of the lambdas with the $M-$country ranking that is obtained when choosing $(\lambda_{12}^*, \lambda_{13}^*, \lambda_{23}^*) = (1,1,1) = \mathbf{1}$ (that is: with the ranking that would be observed under the Strong Focus axiom). This way, we are able to assess the extent to which the 'standard / benchmark / reference' ranking that is obtained under Strong Focus is sensitive to alternative specifications of the parameters governing the trade-offs between meagre and non-meagre attributes. In order to measure the distance between two rankings R, R' of M countries we will use the following function inspired in the work of D'Agostino and Dardanoni (2009):

$$d(R, R') = \frac{1}{(M^3 - M)/3} \sum_{i=1}^{M} (R_i - R'_i)^2 . \tag{5}$$

In equation (5), $R = (R_1, \ldots, R_M), R' = (R'_1, \ldots, R'_M)$ with $R_i, R'_i \in \{1, \ldots, M\}$ are the two rankings of the M countries we are taking into account. The denominator is used to normalize the index between zero and one: the former value is observed whenever $R_i = R'_i \forall i$ and the latter is observed whenever we compare a given ranking $R = (R_1, \ldots, R_M)$ with its opposite ranking $R' = (R'_1, \ldots, R'_M)$ with $R'_i = M - R_i + 1$. The function $d(R, R')$ satisfies various reasonable properties that provide a nice axiomatic characterization[11] . Interestingly, the ranking distance function d is equivalent –up to a monotonic transformation– to the well-known Spearman index of ordinal association ρ: one has that $d = (1-\rho)/2$. Therefore, we can further interpret the values of d in terms of ρ. For instance, when $d = 0, \rho = 1$ (two perfectly correlated rankings are at a distance 0) and when $d = 1, \rho = -1$ (two opposite rankings

[11] These properties are: Monotonicity, Subgroup Consistency, Atomic Monotonicity, Archimedean Property and Minimal Inversion; see D'Agostino and Dardanoni (2009) for details.

are at the maximal possible distance and have the smallest rank correlation). Finally, when $d = 0.5, \rho = 0$ (the distance between two uncorrelated rankings is 0.5).

From now on, for a given $\boldsymbol{\lambda} \in [0,1]^3$ we will denote by $R_{\boldsymbol{\lambda}}$ the $M-$country ranking that is derived from the values of $\boldsymbol{\lambda}$. Therefore, the ranking that is derived under Strong Focus (i.e.: for $\boldsymbol{\lambda} = \mathbf{1}$) will be denoted as $R_{\mathbf{1}}$. In order to assess the sensitivity of the country ranking that is derived under Strong Focus to the choice of alternative $\boldsymbol{\lambda}$, in this paper we will use a function $\delta : [0,1]^3 \to [0,1]$ that for each $\boldsymbol{\lambda}$ computes the value $\delta(\boldsymbol{\lambda}) := d(R_{\boldsymbol{\lambda}}, R_{\mathbf{1}})$. In the empirical section of this paper, we will plot the values of δ associated to the different multidimensional poverty measures shown in Table 1 applied to a set of $M = 54$ countries.

4.2.1 Uncertainty and Robustness

While extremely rich and informative, the use of the ranking distance function $\delta(\boldsymbol{\lambda}) = d(R_{\boldsymbol{\lambda}}, R_{\mathbf{1}})$ has some limitations. To start with, the plot of a three-dimensional function is difficult to visualize, a problem that can be particularly acute when one decides to consider multidimensional poverty measures defined for more than three attributes. Second, $\delta(\boldsymbol{\lambda})$ is a purely descriptive tool that does not take into account the fact that some triples $\boldsymbol{\lambda} \in [0,1]^3$ are more important than others when investigating the robustness of the ranking $R_{\mathbf{1}}$. If a decision-maker is uncertain about the appropriateness of the 'status quo' ranking derived from the choice $\boldsymbol{\lambda} = \mathbf{1}$, she might prefer to introduce some underspecification and consider instead a neighboring set of admissible parameters $\Lambda \subseteq [0,1]^3$ *around* it (that is, with $\mathbf{1} \in \Lambda$). In this framework, the larger the uncertainty regarding the appropriateness of $\boldsymbol{\lambda} = \mathbf{1}$, the larger the size of Λ should be. At the same time, larger sets of admissible parameters Λ are more likely to include triples $\boldsymbol{\lambda}$ with larger values of the ranking distance function $\delta(\boldsymbol{\lambda})$. In this context, a decision maker might be interested in assessing the pace at which the dissimilarity between $R_{\mathbf{1}}$ and a ranking derived from $\boldsymbol{\lambda} \in \Lambda$ increases as the set of admissible parameters Λ that she is willing to accept becomes gradually large. In order to formalize these ideas, we introduce the following notation.

Definition 2. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_8\}$ be the set of vertices of the unit cube $[0,1]^3$. For any $r \in [0,1]$, define $\mathbf{v}_i(r) := r\mathbf{v}_i + (1-r)\mathbf{1}$. Then $\Lambda(r) := C(\{\mathbf{v}_i(r)\}_{1\leq i\leq 8})$, where $C(S)$ is the convex hull of a set S in $\mathbb{R}^3$.

The sets $\Lambda(r)$ correspond to the homothetic expansions of the unit cube $[0,1]^3$ with the vertex $\mathbf{1}$ as the center of homothety[12] . Clearly, $\Lambda(r_1) \subset \Lambda(r_2)$ whenever $r_1 < r_2$. Moreover, $\Lambda(0) = \{\mathbf{1}\}$ and $\Lambda(1) = [0,1]^3$. From now onwards, we will assume that when a decision maker is uncertain about the appropriateness of the ranking $R_{\mathbf{1}}$, she might prefer to consider a set of lambdas belonging to some $\Lambda(r), r \in [0,1]$, where r should be interpreted as the corresponding degree of the decision maker's uncertainty. Whenever a given set of admissible parameters $\Lambda(r)$ is considered, the values of $\delta(\boldsymbol{\lambda})$ (with $\boldsymbol{\lambda} \in \Lambda(r)$) follow a certain distribution with a density function – denoted as $f(r)$ – that will be of interest to the analyst. In particular, it will be interesting to describe $f(r)$ via the corresponding percentiles $p_i(r) := p_i(f(r)) \forall i \in \{1, \ldots 100\}$. By definition, the $p_i(r)$ are bounded between 0 and 1. Finally, it is possible to plot the functions $p_i(r)$ for all $r \in [0,1]$. This gives a picture of the extent to which the set of admissible rankings is different with respect to $R_{\mathbf{1}}$ as we consider larger levels of uncertainty.

[12]Foster, McGillivray and Seth (2013) followed a relatively similar approach in their attempt to assess the robustness of composite index rankings to the choice of alternative weighting schemes belonging to the simplex $\{(w_1, \ldots, w_k) \in \mathbb{R}^k | w_i \geq 0, \sum_i w_i = 1\}$.

5. Empirical illustration

In 2010, the United Nations Development Program (UNDP) presented the values of the so-called 'Multidimensional Poverty Index' (henceforth MPI), which was used to rank more than a hundred countries all over the world. Three main datasets were used to compute the MPI: the Demographic and Health Surveys (henceforth DHS), the Multiple Indicators Cluster Survey and the World Health Survey. The MPI was constructed using the methodology proposed by Alkire and Foster (2011), which –like all multidimensional poverty indices presented in the literature so far– satisfies the Strong Focus axiom. In this section, we will investigate the robustness of different multidimensional poverty indices inspired in UNDP's MPI when relaxing the Strong Focus assumption in favor of the weak version of the axiom[13] . More specifically, we will use the indices $P(G, E)$ characterized in Theorems 1 and 2, which generalize the existing multidimensional poverty measures presented in the literature so far (recall Remark 3). For that purpose, we will work with the same countries appearing in the official UNDP's MPI list but restricting our attention to those countries whose MPI values were estimated using the Demographic and Health Surveys (totalling $M = 54$ countries[14]). This way, we avoid the comparability problems that might arise if we used alternative data sources (a problem that actually afflicts the official MPI values).

[13]It should be highlighted that the multidimensional poverty indices presented in this section are *not* exactly the same as the official UNDP's MPI –even if some of them are quite similar, see Figure (4) below– so their values are not strictly comparable. However, this is not a problem for the main goal of this section, to wit, test the sensitivity of poverty rankings when the Strong Focus axiom is relaxed in favor of Weak Focus.

[14]The 54 countries included in the dataset and the year in which the DHS was taken are: Albania (2009), Armenia (2005), Azerbaijan (2006), Bangladesh (2007), Benin (2006), Bolivia (2008), Cambodia (2005), Cameroon (2004), Colombia (2010), Congo (2009), Democratic Republic of the Congo (2007), Côte d'Ivoire (2005), Dominican Republic (2007), Egypt (2008), Ethiopia (2005), Gabon (2000), Ghana (2008), Guinea (2005), Guyana (2005), Haiti (2006), Honduras (2006), India (2005), Indonesia (2007), Jordan (2009), Kenya (2009), Lesotho (2009), Liberia (2007), Madagascar (2009), Malawi (2004), Maldives (2009), Mali (2006), Republic of Moldova (2005), Mozambique (2009), Namibia (2007), Nepal (2006), Nicaragua (2006), Niger (2006), Nigeria (2008), Pakistan (2007), Peru (2004), Philippines (2008), Rwanda (2005), Sao Tome and Principe (2009), Senegal (2005), Sierra Leone (2008), Swaziland (2007), Tanzania (2008), Timor-Leste (2009), Turkey (2003), Uganda (2006), Ukraine (2007), Viet Nam (2002), Zambia (2007) and Zimbabwe (2006).

The DHS are nationally representative surveys with large sample sizes and questionnaires that are virtually identical across time and countries. In most surveys, households are selected based on a standard stratified and clustered design, and, within the household, one woman, aged 15-49, is selected at random as the focus of the interview. In addition, all living children up to a given age (usually 60 months, but sometimes 36 months) born to that woman are weighed and measured. These surveys have been widely used by different researchers to measure poverty levels in developing countries (see, for instance, Sahn and Stifel 2000 or Duclos, Sahn and Younger 2006). Following the approach used in the construction of the MPI, the basic unit of our analysis will be the household (totalling $876,742$ observations among the 54 surveys). Even if this choice is not ideal and is somewhat driven by data constraints, it is intuitive and facilitates comparisons with the original MPI.

5.1 Definition of dimensions, indicators and thresholds

There are many potential well-being variables available in the DHS that can be used to measure deprivation in a multiattribute framework. Mimicking the methodology used in the definition of UNDP's MPI, here we concentrate on three dimensions: Education, Health and Standard of Living.

In the education dimension we will use the indicator 'Years of Schooling', which acts as a proxy for the level of knowledge and understanding of household members. While this indicator has different shortcomings (*e.g.*: does not capture quality of education or level of knowledge attained), it is a robust and widely available indicator that provides the closest feasible approximation to levels of education for household members. In order to measure households' deprivation in terms of 'Years of Schooling', we will focus on the highest value of that variable among the corresponding household members. While other approaches could have been easily implemented as well (*e.g.*: averaging the variable across household members), we have preferred to use the highest value of the variable as representative of the household education status for the following reasons. As Basu and Foster (1998) –and many others after them– suggest, all household members benefit from the abilities of an educated person in the household, regardless of each person's actual level of education. In addition, this has been the approach followed in the construction of MPI's education component (in that context, all household members are considered non-deprived if at least one person has a high level of education). In order to determine whether a given household is deprived or not in the education dimension, we set the poverty threshold at five years of education (*i.e.*: $z_1 = 5$). This is the threshold that has been used in the construction of the MPI. Analogously, recent studies without literacy information have used this threshold as a proxy to classify individuals as literates or illiterates (*e.g.*: Grimm et al. 2008). The upper bound of this variable –which is needed to compute the excess gaps e_{ij}– is set at 20 years (i.e.: $U_1 = 20$).

As acknowledged by different authors, health is the most difficult dimension to measure in the assessment of multidimensional poverty because of the lack of appropriate data. Mimicking the MPI methodology, we use information on the nutritional status of individuals to estimate deprivations in the health dimension. Adults who are malnourished are susceptible of different health disorders, they are less able to concentrate and learn and may not perform as well in work (Alkire and Santos 2010:12). The indicator that will be used to assess individuals' malnutrition is the Body Mass Index (BMI), which is defined as the ratio between weight (measured in kilograms) and the square of height (measured in meters)[15] .

[15]As is known, the MPI also includes information on child nutrition. However, that information is missing in many households and its inclusion renders comparisons between households with and without children more problematic on conceptual grounds. For these reasons, we have preferred not to include that indicator in our assessment of multidimensional poverty.

As is well known, this is an imperfect indicator with some limitations: (i) It does not reflect micronutrient deficiencies, (ii) Individuals' nutritional status is not always related to poverty, it might be influenced by alimentary disorders, fashion norms or recent illnesses. Like the MPI, we do not consider the problem of obesity. When it comes to measure households' deprivation in terms of malnutrition, we have chosen the lowest BMI among all household members[16] . This criterion is analogous to the one used in the construction of the MPI – where all household members are considered to be deprived in nutrition if at least one under-nourished person is observed in the household. It is a widely common practice to establish the BMI threshold to determine whether individuals are malnourished or not at a value of 18.5 (that is: $z_2 = 18.5$). At the other extreme, we have truncated the BMI distribution from above at the value of 25. When the BMI is larger than 25 individuals are considered to be overweight, so larger values of the index are not expected to be beneficial for their well-being. Therefore, U_2 is set at 25.

Lastly, our indices also include a standard of living component. Since the DHS were not designed for economic analysis, there are no data on income or expenditures–the standard money metric measures of standard of living. Despite this drawback, the DHS do contain information on household assets that can be employed to represent an alternative to a money metric. In the absence of income or expenditure data, we derive a welfare index constructed from the households' asset information available in the DHS. Asset indices have been widely used in the literature (e.g.: Filmer and Pritchett 2001, Sahn and Stifel 2000, 2003, Grimm et al 2008, Harttgen and Klasen 2011) and their advantages and disadvantages are well known (Filmer and Scott 2012 provide an excellent survey in this regard)[17] . Asset indices are

[16]Again, it could *a priori* be possible to implement other summary measures like some BMI average across household members. However, the compensations involved in this averaging process might fail to detect malnourished individuals.

[17]On the negative side, different authors have emphasized that: (i) Being discrete functions, there might

defined as a weighted linear sum $\sum_i w_i a_{hi}$, where w_i is the weight attached to asset 'i' and $a_{hi} \in \{0, 1\}$ refer to the absence or presence of asset 'i' in household 'h'. In this paper, the asset index has been constructed with 13 items[18] and with equal weights ($w_i = 1/13 \forall i$). While some authors use Factor Analysis or Principal Components techniques to derive the corresponding weights (e.g.: Filmer and Pritchett 2001, Sahn and Stifel 2000, 2003, Harttgen and Klasen 2011), we have preferred to keep the equal weighting scheme as already done by many others (e.g.: Montgomery et al 2000, Case, Paxson and Ableidinger 2004, Hohmann and Garenne 2010, Permanyer 2013) for the sake of simplicity and transparency. In addition, this is also the approach implicitly followed in the construction of the MPI. In the related literature, it is common to draw the poverty threshold for the standard of living distribution at the $25^{th}, 33^{rd}$ or the 40^{th} percentiles (*e.g.*: Sahn and Stifel 2000, 2003). In this paper we report the values corresponding to the 33^{rd} percentile (the conclusions remain essentially the same for the other cutoffs). The upper bound of the standard of living distribution that is needed to compute the excess gaps equals $U_3 = 1$ (this is the maximal possible value of the asset index for a household owning all assets included in our list).

be the risk that observations are clustered around certain values, therefore posing a challenge to the task of estimating the underlying welfare distribution; (ii) The list of assets included in these indices typically refers to basic commodities that do not cover many of the goods and services that are generally available to high-income households; (iii) Asset indices have been criticized because they might not correctly capture differences between urban and rural areas. On the positive side, it is acknowledged that: (i) The reporting of household assets is less vulnerable to measurement errors than the reporting of income or expenditures; (ii) Asset indices might be a better proxy for long-term living standards than current income because they are less vulnerable to economic shocks and fluctuations over time than income or expenditure. To sum up, even if their values should be taken with caution, asset indices seem a viable – though imperfect – way of assessing material welfare.

[18]The list of assets used in this paper is the following: 1. Electricity: The household has electricity; 2. Sanitation (toilet facility): The household sanitation facility is improved and not shared with other households; 3. Water: the household does have access to clean drinking water, or clean water is less than 30 minutes walking from home; 4. Floor: The household has no dirt, sand or dung floor; 5. Roof: The household has finished roofing; 6. Walls: The household has finished walls; 7. Cooking fuel: The household does not cook with dung, wood or charcoal; 8. Radio: The household has a Radio; 9. TV: The household has a TV; 10. Telephone: The household has a Telephone; 11. Refrigerator: The household has a Refrigerator; 12. Bike: The household has a Bike; 13. Motor vehicle: The household has a Motor vehicle (Motorbike, Car, Truck).

Summing up, the indicators used in the construction of our multidimensional poverty indices are essentially the same as those used in the MPI. However, while the MPI is constructed following an "ordinal approach" (that is: what matters when computing its values is whether households are deprived or non-deprived in the corresponding dimensions), the indices presented here need cardinal information to compute deprivation and excess gaps in a meaningful way.

5.2 Empirical results

Before carrying out our tests to assess the robustness of MPI-like rankings, we have performed a validation check using external data to assess the quality and soundness of the 54-country dataset we have created for the empirical section of this paper. More specifically, we have compared the official UNDP's 2011 MPI ranking –restricted to the 54 countries whose MPI values where estimated using DHS – with the ranking we have obtained using the Alkire and Foster (2011) index[19] applied to the dataset described in the previous section (see Figure (4)). As can be seen, both measures tend to rank countries in a strongly linear fashion: the rank correlation coefficient equals 0.96 and the ranking distance function $d(R, R')$ corresponding to those rankings (see equation (5)) is as low as 0.02. Therefore, we can be reasonably confident that the dataset we will be working with tends to rank countries in a very similar way as the official MPI does whenever we restrict our attention to the Strong Focus axiom.

We will now present results regarding the robustness of the 54-country rankings derived from the multidimensional poverty measures shown in Table 1 (*i.e.*: $P^{AF}, P^{BC}, P^{T1}, P^{T2}, P^{T3}$ and P^{T4}) to alternative specifications of the Weak Focus axiom[20] . For that purpose, we

[19] We have used the P^{AF} index (see Table 1) with $\alpha = 2$ and using the union approach as identification method.

[20] Recall that the index P^W is essentially the same as P^{T2}, so it has not been included in the list.

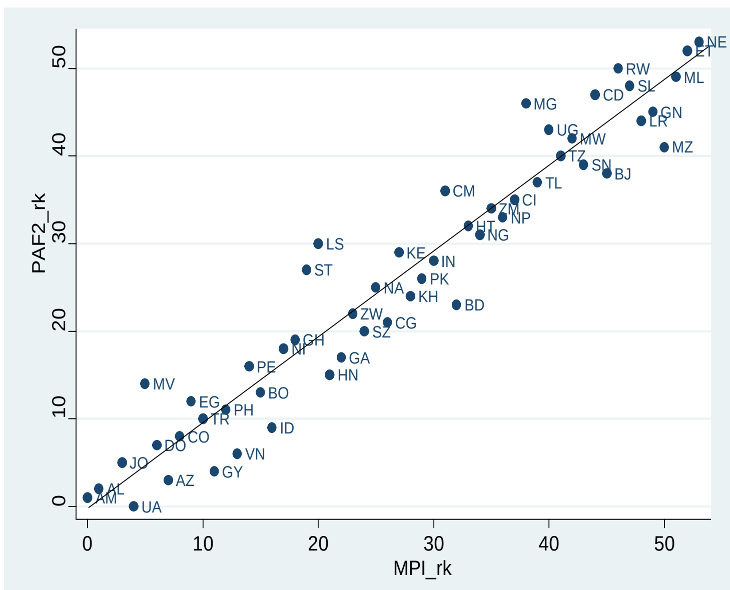

Figure 4: Comparison of country rankings arising from the values of UNDP's MPI and P^{AF} with $\alpha = 2$ (see Table 1) for 54 countries with Demographic and Health Surveys. Country labels follow the ISO-3166 coding scheme. The solid line line is the 45^o equality line drawn for comparative purposes. Source: authors' calculations using UNDP and DHS data.

will work with the generalized version of those measures characterized in Theorems 1 and 2 (i.e.: the corresponding $P(G,E)$ indices shown in equations (3) and (4) with the appropriate functions ψ, φ_{jl} and parameter θ–see footnote #9) and allow the parameters governing trade-offs between meagre and non-meagre attributes (i.e.: the lambdas $\boldsymbol{\lambda} = (\lambda_{12}, \lambda_{13}, \lambda_{23})$) to take any value within their domain. In this context, λ_{12} (resp. λ_{13} and λ_{23}) corresponds to the parameter governing trade-offs between education and health (resp. trade-offs between education and wealth and trade-offs between health and wealth). As explained above, the corresponding 'reference' or 'status quo' ranking is the one that is obtained from the values of the indices $P^{AF}, P^{BC}, P^{T1}, P^{T2}, P^{T3}$ and P^{T4} themselves (that is: the indices satisfying the Strong Focus axiom, which correspond to the choice of $\boldsymbol{\lambda} = (1,1,1)$). For each of the multidimensional poverty indices considered in this paper, Figure (5) plots the values of the corresponding 'reference ranking distance function' $\delta(\boldsymbol{\lambda})$ for a wide range of values of $\boldsymbol{\lambda} \in [0,1]^3$. More specifically, we plot the values of the corresponding $\delta(\boldsymbol{\lambda})$ for all possible values of λ_{13} and λ_{23} whenever $\lambda_{12} \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. In order to interpret this graph, recall that whenever the values of $\delta(\boldsymbol{\lambda})$ are "around" 0.5 (as a rule of thumb: between 0.4 and 0.6), the Strong Focus ranking and the Weak Focus ranking derived from the values of $\boldsymbol{\lambda}$ are essentially uncorrelated. When this happens, the rankings derived from both axioms look as if they were independently generated.

As can be seen in Figure (5), the multidimensional poverty rankings that are obtained from the Weak Focus axiom are, in general terms, quite different with respect to the reference rankings derived from Strong Focus. As expected, the values of $\delta(\boldsymbol{\lambda})$ tend to be lower for the layers $\lambda_{12} = 0$ or $\lambda_{12} = 0.2$ (that is, when λ_{12} is 'small') for all multidimensional poverty indices studied in this paper. However, even in those layers the values of $\delta(\boldsymbol{\lambda})$ are considerably large in a close vicinity of $\boldsymbol{\lambda} = (1,1,1)$. These results suggest that even when very small room for interaction is allowed between deprived and non-deprived attributes, our assessments of multidimensional poverty can differ to a large extent.

Interestingly, the patterns of variation of the $\delta(\boldsymbol{\lambda})$ function within the unit cube are highly non-linear, so they are quite difficult to describe. This is particularly the case for the poverty measure $P(G,E)$ associated to P^{T1}: in that case the values of $\delta(\boldsymbol{\lambda})$ vary wildly in a quite unpredictable way. In fact, the highest reported value of $\delta(\boldsymbol{\lambda})$ $(= 0.68)$ has been observed for that specific index. At the other extreme, we find other indices for which the behavior of $\delta(\boldsymbol{\lambda})$ looks somewhat more smooth and parsimonious. This is the case for the poverty measures $P(G,E)$ associated to P^{T2}, P^{T3} and, to a lesser extent, P^{BC}. Somewhere in between, we can find the poverty measures $P(G,E)$ associated to P^{AF} and P^{T4}. In general, the highly complex behavior of $\delta(\boldsymbol{\lambda})$ might be attributable to the multiplicative functional form of the correction functions accompanying the deprivation gaps g_{ij} of equation (3).

Inspecting Figure (5), one might want to investigate the behavior of $\delta(\boldsymbol{\lambda})$ 'near' $\boldsymbol{\lambda} = \mathbf{1}$ in more detail. A decision maker who is uncertain about the appropriateness of the Strong Focus axiom might prefer to allow for a certain degree of underspecification and let the lambdas free within the admissible sets $\Lambda(r) \subseteq [0,1]^3$ around $\mathbf{1} \in \Lambda(r)$ (where the values of r are interpreted as the degree of the decision maker's uncertainty). In Figure (6), we plot the percentiles $p_i(r)$ of the $\delta(\boldsymbol{\lambda})$ distribution when the values of $\boldsymbol{\lambda}$ are restricted to $\Lambda(r)$ for the different values of $r \in [0,1]$ and for each the different multidimensional poverty measures investigated in this paper. As expected, virtually all percentiles $p_i(r)$ increase as we increase the values of r. In other words: as we gradually enlarge the size of admissible sets $\Lambda(r)$, the set of admissible rankings becomes increasingly different with respect to the status quo ranking that prevails under the Strong Focus axiom[21] . As can be seen, the values reached

[21] Note that, *a priori*, this should not always be necessarily the case.

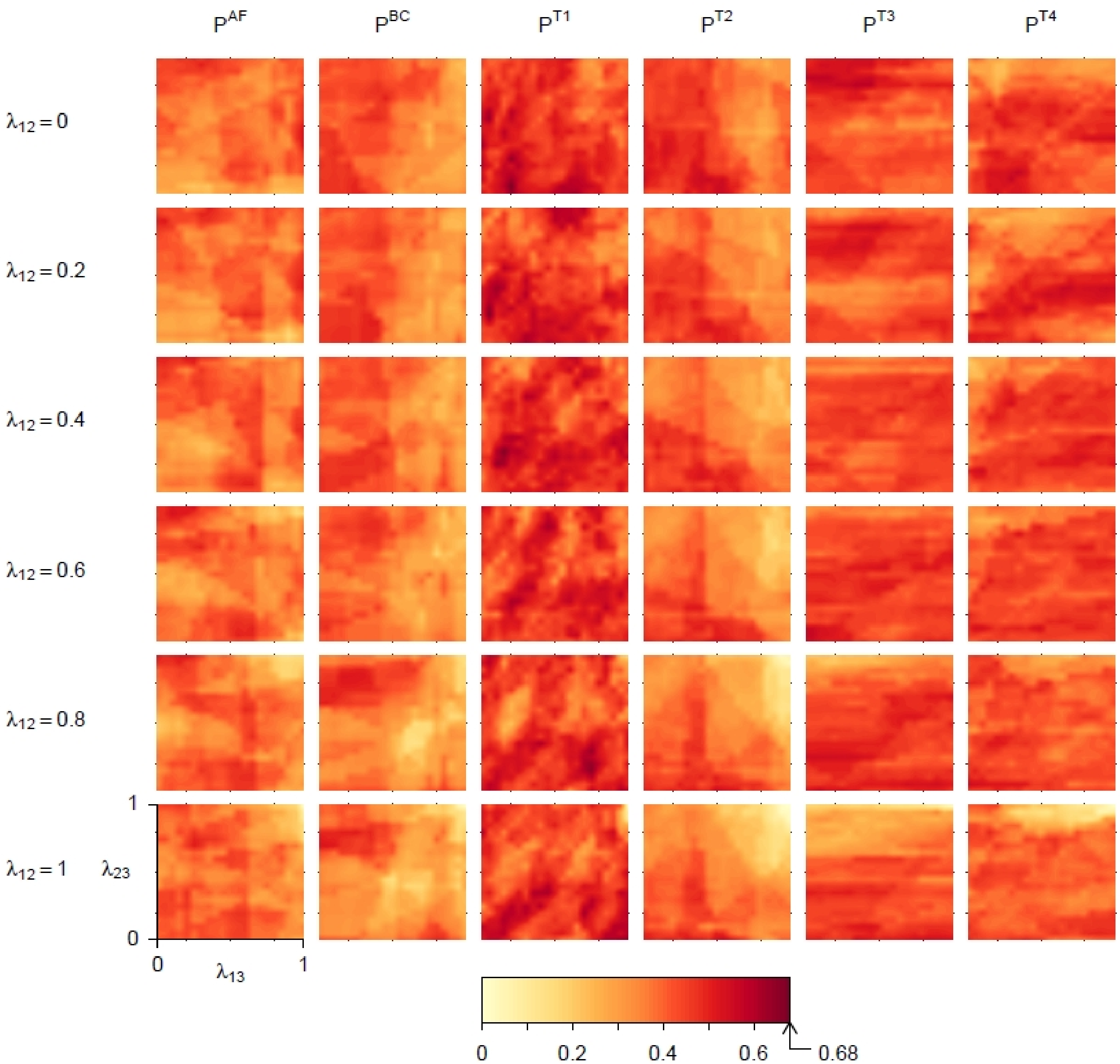

Figure 5: Values of $\delta(\boldsymbol{\lambda})$ for the multidimensional poverty indices $P(G, E)$ characterized in this paper. Each column is labeled according to the specific poverty index for which the corresponding $\delta(\boldsymbol{\lambda})$ values are calculated. Each row corresponds to different values of $\lambda_{12} \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. The squares within this graph have been plotted using a linear interpolation function over a regular grid of 51×51 test points for the different values of λ_{13} and λ_{23} between 0 and 1. Source: Author's calculations using DHS data.

by those distributions is remarkably high. For the cases of P^{T1}, P^{T3} and P^{T4} the median $p_{50}(r)$ soon approaches – and in some cases crosses – the threshold of 0.4, signifying that for those indices most admissible rankings are virtually uncorrelated with respect to the reference ranking $R_{\mathbf{1}}$. For P^{AF}, P^{BC} and P^{T2} the $p_{50}(r)$ values tend to be smaller but are substantially large as well (in many cases hovering between 0.3 and 0.4).

Of particular interest is the pace at which the percentiles $p_i(r)$ increase with r, specially when r is close to 0 – as this measures the sensitivity of the reference ranking $R_{\mathbf{1}}$ to a slight weakening of the Strong Focus axiom. Most percentiles shown in Figure (6) tend to increase at a marginally decreasing pace, but the degree of concavity is quite different across indices. At one extreme we have the case of P^{T2}: for values of r between 0 and 0.1, the corresponding percentiles barely increase above 0, and when r goes beyond 0.1, those percentiles tend to slowly increase at a fairly constant pace. At the other extreme, we find the cases of P^{T1}, P^{T3} and P^{T4}: for those indices, slight increases of r lead to remarkable jumps in the corresponding percentile functions, which seem to stabilize when r goes beyond 0.2. Therefore, the rankings derived from these indices are extremely sensitive to the values of $\boldsymbol{\lambda}$ even in a close vicinity of $\mathbf{1}$. In between, we find the cases of P^{AF} and P^{BC}, which also increase at a marginally decreasing pace.

Summing up, the different poverty measures studied here react in a quite different way when over-achievements are allowed to intervene in the assessment of individuals' poverty levels. While some of these measures are much more sensitive than others, all of them bring to light a completely different assessment of the extent of multidimensional poverty when we gradually abandon the rigidity of the Strong Focus axiom.

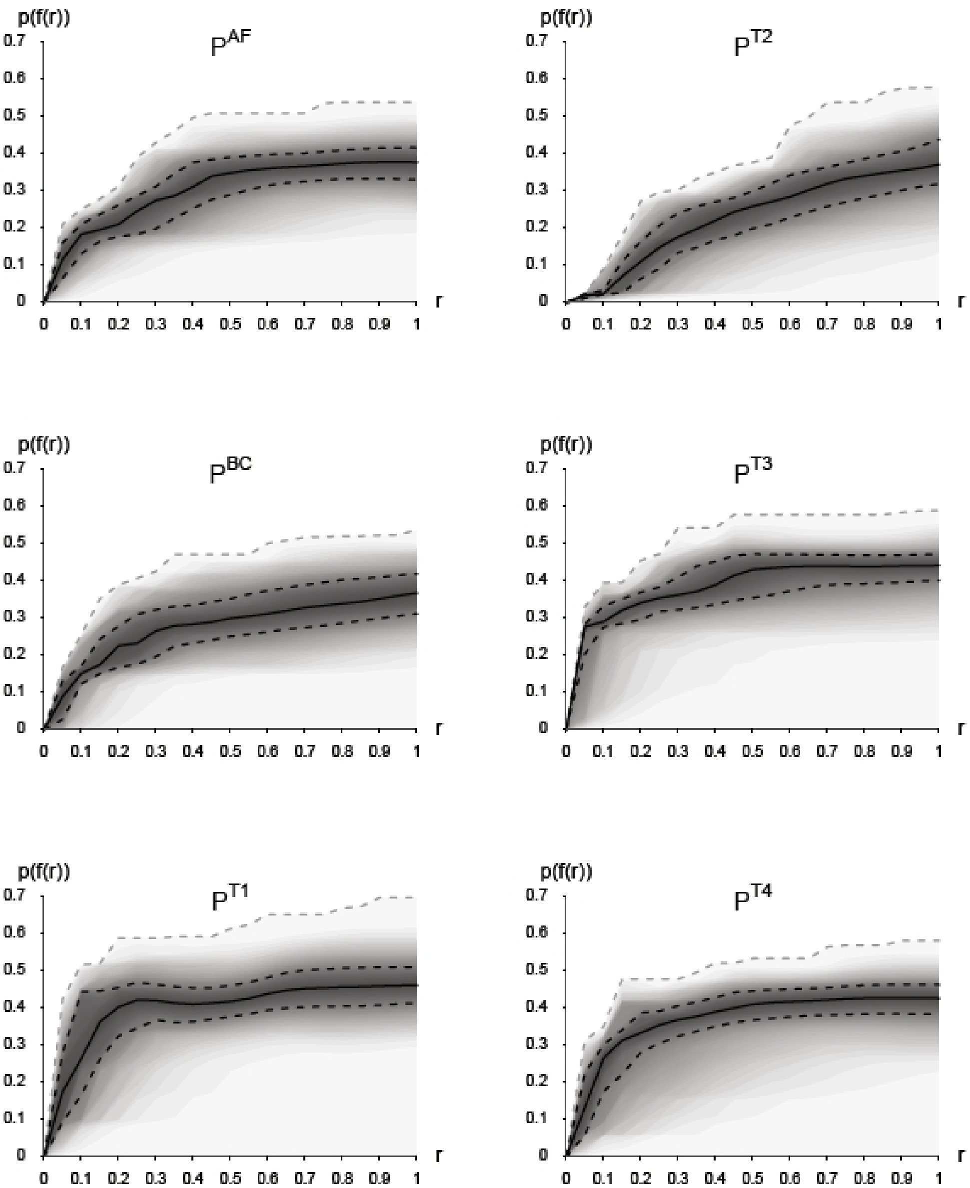

Figure 6: Percentiles of the $\delta(\boldsymbol{\lambda})$ distribution when $\lambda \in \Lambda(r) \forall r \in [0,1]$ corresponding to the multidimensional poverty indices $P(G,E)$ characterized in this paper. The black solid lines represent the values of the 50^{th} percentile $p_{50}(r)$. The black (resp. grey) dashed lines represent the 25^{th} and 75^{th} percentiles (resp. the 100^{th} percentile). Source: Author's calculations using DHS data.

6. Discussion and concluding remarks

The Focus axiom is one of the cornerstones upon which traditional income poverty measures are based. It basically states that an income poverty measure should not be sensitive to the distribution of incomes above the poverty line. However, after the recent introduction of the multidimensional setting with different dimensions and poverty lines, this axiom can be stated in different ways. A strong version of this axiom (known as Strong Focus) states that a poverty measure should be insensitive to the increase of non-deprived attributes of any individual. A weaker version of this axiom (known as Weak Focus) states that, after identifying who is poor and who is not, a poverty measure should be insensitive to the increase of attributes of non-poor individuals only. Interestingly, all multidimensional poverty measures introduced in the literature so far satisfy the stronger version of the axiom, but this seems to be unduly restrictive since it precludes the possibility that over-achievements in non-deprived dimensions can influence and somehow compensate deprivation levels in other dimensions for those individuals identified as 'poor'. In this paper we have argued that this is an interesting and plausible possibility that can have potentially relevant implications for the conceptualization and measurement of multidimensional poverty, so it should be seriously taken into account when evaluating the poverty levels in different societies.

The main reason why different researchers are reluctant to relax the Strong Focus axiom is that they do not want an increase in non-deprived attributes only to be able to pull individuals out of poverty (see Alkire and Foster 2011:481). In line with the stylized setting shown in Figure 1, these researchers reasonably claim that an individual D with very good health but with no income whatsoever (e.g.: an old street beggar who is a pavement dweller) should be considered as 'poor' besides his high life expectancy. As a matter of fact, this is an important problem that is encountered by some of the multidimensional poverty indices suggested by Lugo and Maasoumi (2008). According to some of the poverty measures presented in that paper and given the assumed tradeoffs across dimensions, it is possible for a poor person to be lifted out of poverty as a result of an increment in a nondeprived dimension, thus lowering the measured level of poverty[22] . In this discussion, it is important to point out that relaxing the Strong Focus axiom in favor of its weak version does not necessarily imply that individuals can be pulled out of poverty by increases in non-deprived dimensions. In the approach presented in this paper, an individual that is identified as 'poor' does not change its status even if the achievement levels in non-deprived attributes are arbitrarily increased (the only thing that is allowed to change is the 'intensity' of poverty of that individual, not its poor/non-poor status). Put in other words: our approach does not lift individual D out of poverty because of his high life expectancy, but it introduces the possibility of asserting that he is not as poor as another individual E who is also a street beggar but whose health is on the corresponding health poverty line (typically a relatively bad health status).

It is important to emphasize that the approach presented here transcends the hotly debated and still unresolved problem of 'identification' of the poor (see footnote #1). The methodology introduced here does not take a particular stance in that problem but rather suggests that once a poor individual has been identified –no matter how–, the dimensions in which she is not deprived might be allowed to somehow influence her deprivation level[23] . Interestingly, this idea is reminiscent of the notion of 'poverty reduction failure' which has

[22]In that paper, the authors present three alternative multidimensional poverty measures, but two of them satisfy the Strong Focus axiom. The only measure violating Strong Focus and satisfying Weak Focus uses the so-called 'poverty frontier' approach, which basically generates a composite index of well-being so that poverty can be defined in terms of that single-dimensional index. It is in this context that over-achievements in non-deprived dimensions can pull individuals out of poverty. The use of the 'poverty frontier' approach has been criticized by Bourguignon and Chakravarty (2003:27-28) for being a methodology that looses track of the dimension-specific poverty gaps–therefore not being a truly multidimensional method–so it has not been explored in this paper.

[23]

been recently introduced in the literature (see, for instance, Kanbur and Mukherjee 2007, Ravallion 2010 or Thori and Moene 2011), albeit it is quite different in nature. On the one hand, in those papers poverty is conceptualized on the basis of income distributions alone (*i.e.*: it is unidimensional). On the other hand, the main goal in that literature is to assess the extent of poverty relative to the resources available in the society to eradicate it, which is a quite different exercise from our assessment of deprivation in a multidimensional framework.

Other things being equal, allowing for trade-offs between meagre and non-meagre attributes inevitably reduces the extent of overall multidimensional poverty *levels*. This in itself does not necessarily imply lower incidence of poverty or increased satisfaction with the status quo. The approach suggested here offers a way of gauging actual poverty more accurately and differentiates between otherwise indistinguishable comparisons. In this respect, the empirical results shown in this paper suggest that our assessments of multidimensional poverty can differ dramatically when the overly restrictive Strong Focus is abandoned in favor of weaker versions of the axiom.

While the methodology presented here allows modelling different degrees of complementarity / substitutability between couples attributes – a substantial improvement with regard to the current state of the literature that fixes those degrees at a constant level for all possible couples – much work still remains to be done to determine their values. In this regard, the robustness techniques suggested in this paper can be seen as a preliminary but exhaustive approach that should certainly be refined in future research.

In case one used the intersection approach to identify 'the poor', the ideas introduced in this paper would not be applicable because poor individuals would not have any achievements above any dimension-specific poverty line.

7. Appendix

Proof of Theorem 1. Since P satisfies *Subgroup Decomposability*, it can be written as

$$P = \frac{1}{n}\sum_{i=1}^{n} p(\mathbf{g}_i, \mathbf{e}_i)$$

for some function $p : R^k \times [0,1]^k \to \mathbb{R}$. Clearly, for any $(\mathbf{g}, \mathbf{e}) \in R^k \times [0,1]^k$, $p(\mathbf{g}, \mathbf{e}) = P(G, E)$ for some deprivation and achievement matrices $(G, E) \in (\mathcal{G}_S^{n\times k}, \mathcal{E}_S^{n\times k})$ where all rows are equal to $(\mathbf{g}, \mathbf{e})$. Therefore, since P satisfies *Subgroup Decomposability, Continuity, Homotheticity, Weak Dimension Separability, Monotonicity on Deprivation Gaps* and *Independence*, p will satisfy them too. Following Lasso de la Vega and Urrutia (2011:190), it can be shown that *Monotonicity on Deprivation Gaps* implies minimal increasingness and strict essentiality (see Blackorby and Donaldson 1982:251). Moreover, the domain of p is $R^k \times [0,1]^k$, which is connected and topologically separable. In an analogous way to Blackorby and Donaldson (1982: 252), based on Gorman (1968:369) and Blackorby, Primont and Russel (1978:127) it can be shown that p is additively separable and can be written as

$$p(\mathbf{g}, \mathbf{e}) = h^{\circ}\left(\sum_{j=1}^{k} p_j(g_j, \mathbf{e})\right) \tag{6}$$

where h° and $p_j, j \in \{1, \ldots, k\}$ are continuous real-valued functions and h° is increasing. By *Homotheticity*, p can be written as

$$p(\mathbf{g}, \mathbf{e}) = \psi\left(\overline{p}(\mathbf{g}, \mathbf{e})\right) \tag{7}$$

where ψ is an increasing function and $\overline{p}$ is linearly homogeneous on $\mathbf{g}$. Combining equations (6) and (7) one has that

$$\overline{p}(\mathbf{g}, \mathbf{e}) = \psi^{-1}\left(h^{\circ}\left(\sum_{j=1}^{k} p_j(g_j, \mathbf{e})\right)\right) = p^{*}\left(\sum_{j=1}^{k} p_j(g_j, \mathbf{e})\right) \tag{8}$$

for some increasing function $p^* := \psi^{-1} \circ h^\circ$. Using equation (8), for each $l \in \{1, \ldots, k\}$ we can define the functions

$$h_l(g_l, \mathbf{e}) := \overline{p}(g_{\min}, \ldots, g_{\min}, g_l, g_{\min}, \ldots, g_{\min}, \mathbf{e}) = p^* \left(p_l(g_l, \mathbf{e}) + \sum_{j \neq l} p_j(g_{\min}, \mathbf{e}) \right). \quad (9)$$

Since $\overline{p}$ is linearly homogeneous on $\mathbf{g}$, so are the functions $h_l(g_l, \mathbf{e})$. Therefore

$$h_l(\lambda g_l, \mathbf{e}) = \lambda h_l(g_l, \mathbf{e}) \quad (10)$$

for any $\lambda \in R$ and for all $l \in \{1, \ldots, k\}$. As a consequence, there exist continuous functions $\phi_l(\mathbf{e})$ such that

$$h_l(g_l, \mathbf{e}) = g_l h_l(1, \mathbf{e}) = g_l \phi_l(\mathbf{e}) \quad (11)$$

Plugging equations (9) and (11) we have that

$$g_l \phi_l(\mathbf{e}) = p^* \left(p_l(g_l, \mathbf{e}) + \sum_{j \neq l} p_j(g_{\min}, \mathbf{e}) \right). \quad (12)$$

Hence

$$p_l(g_l, \mathbf{e}) = p^{*-1} \left(g_l \phi_l(\mathbf{e}) \right) - \sum_{j \neq l} p_j(g_{\min}, \mathbf{e}) \quad (13)$$

Substituting equation (13) in equation (8), one has that

$$\overline{p}(\mathbf{g}, \mathbf{e}) = p^* \left(\sum_{j=1}^{k} \left[p^{*-1} \left(g_j \phi_j(\mathbf{e}) \right) - \sum_{l \neq j} p_l(g_{\min}, \mathbf{e}) \right] \right) = p^* \left(\sum_{j=1}^{k} p^{*-1} \left(g_j \phi_j(\mathbf{e}) \right) + \varsigma(\mathbf{e}) \right) \quad (14)$$

for some continuous functions $\phi_j(\mathbf{e}), \varsigma$ and a continuous increasing function p^*. Fixing any $\mathbf{e} \in [0, 1]^k$, equation (14) is essentially the same as equation (34) in Blackorby and Donaldson (1982:260). Therefore, following those authors–who in turn draw from Eichhorn (1978:32-34)– it can be proven that $p^{*-1} =: f$ must satisfy the following functional equation

$$f(\lambda u) = \alpha(\lambda) f(u) + b(\lambda) \quad (15)$$

Without the domain restrictions on λ and u, the solutions to equation (15) are well-known (Aczel et al 1986). It is straightforward to show that the solution for equation (15) on the present restricted domain is

$$f(u) = \left\{ \begin{array}{c} au^{\theta} + b \\ \\ c\ln(u) + d \end{array} \right\} \tag{16}$$

for some parameters a, b, c, d, θ (with $\theta \neq 0$). Since continuity of f at 0 precludes the logarithmic solution, the general solution of equation (14) can be written as

$$\overline{p}(\mathbf{g}, \mathbf{e}) = p^{*} \left(\sum_{j=1}^{k} a \left(g_j \phi_j(\mathbf{e}) \right)^{\theta} + kb + \varsigma(\mathbf{e}) \right) = \left(\sum_{j=1}^{k} \left(g_j \phi_j(\mathbf{e}) \right)^{\theta} + \frac{b(k-1) + \varsigma(\mathbf{e})}{a} \right)^{1/\theta}. \tag{17}$$

Since $\overline{p}$ is linearly homogeneous on $\mathbf{g}$, one must have that $\left(b(k-1) + \varsigma(\mathbf{e})\right)/a = 0$. Therefore, equation (7) can be rewritten as

$$p(\mathbf{g}, \mathbf{e}) = \psi \left(\left(\sum_{j=1}^{k} \left(g_j \phi_j(\mathbf{e}) \right)^{\theta} \right)^{1/\theta} \right) \tag{18}$$

for some continuous increasing function ψ.

We will now decompose the continuous functions $\phi_j(\mathbf{e})$. Inspecting Table 1, we see that the values of $g_{\min}$ can be either 0 or 1.

Case 1. Assume $g_{\min} = 0$.

Consider now a hypothetical scenario *where all individuals have exactly the same achievement distribution*: they are all deprived in exactly the same dimension and non-deprived in the other ones. Denote that specific dimension by d. Therefore, $\mathbf{g} := (g_{\min}, \ldots, g_{\min}, g_d, g_{\min}, \ldots, g_{\min}) = (0, \ldots, 0, g_d, 0, \ldots, 0)$. In that case, equation (18) can be written as

$$p(\mathbf{g}, \mathbf{e}) = \psi \left(g_d \phi_d(\mathbf{e}) \right) \tag{19}$$

Define the excess vector $\boldsymbol{\varepsilon}_0 = (0, \ldots, 0)$. Plugging $\boldsymbol{\varepsilon}_0$ into equation (19) yields

$$p(\mathbf{g}, \boldsymbol{\varepsilon}_0) = \psi\left(g_d \phi_d(0, \ldots, 0)\right) = \psi(c g_d) \tag{20}$$

for some constant $c \in \mathbb{R}$. Consider now the excess vector $\boldsymbol{\varepsilon}_1 = (e_1, 0, \ldots, 0)$. Plugging $\boldsymbol{\varepsilon}_1$ into equation (19) yields

$$p(\mathbf{g}, \boldsymbol{\varepsilon}_1) = \psi\left(g_d \phi_d(e_1, 0, \ldots, 0)\right) = \psi(g_d \varphi_{d1}(e_1)) \tag{21}$$

for some continuous function φ_{d1} defined on $[0, 1]$. Defining $\widetilde{\mathbf{g}} = (0, \ldots, 0, \left(g_d \varphi_{d1}(e_1)\right)/c, 0, \ldots, 0)$ and using equations (20) and (21), it turns out that

$$p(\widetilde{\mathbf{g}}, \boldsymbol{\varepsilon}_0) = \psi(g_d \varphi_{d1}(e_1)) = p(\mathbf{g}, \boldsymbol{\varepsilon}_1). \tag{22}$$

Define now $\boldsymbol{\varepsilon}_{12} = (e_1, e_2, 0, \ldots, 0)$ and $\boldsymbol{\varepsilon}_2 = (0, e_2, 0, \ldots, 0)$. Applying *Independence* to equation (22) yields

$$p(\mathbf{g}, \boldsymbol{\varepsilon}_{12}) = p(\widetilde{\mathbf{g}}, \boldsymbol{\varepsilon}_2). \tag{23}$$

According to equation (19),

$$p(\mathbf{g}, \boldsymbol{\varepsilon}_{12}) = \psi\left(g_d \phi_d(\boldsymbol{\varepsilon}_{12})\right) \tag{24}$$

and

$$p(\widetilde{\mathbf{g}}, \boldsymbol{\varepsilon}_2) = \psi\left(\left(\left(g_d \varphi_{d1}(e_1)\right)/c\right) \phi_d(\boldsymbol{\varepsilon}_2)\right) = \psi\left(\left(\left(g_d \varphi_{d1}(e_1)\right)/c\right) \varphi_{d2}(e_2)\right) \tag{25}$$

for some continuous function φ_{d2} defined on $[0, 1]$. Plugging (24) and (25) into (23) yields

$$\psi\left(g_d \phi_d(\boldsymbol{\varepsilon}_{12})\right) = \psi\left(\left(\left(g_d \varphi_{d1}(e_1)\right)/c\right) \varphi_{d2}(e_2)\right). \tag{26}$$

Since ψ is a continuously increasing function, ψ^{-1} is well defined. Applying ψ^{-1} to both sides of equation (26) yields

$$\phi_d(\boldsymbol{\varepsilon}_{12}) = (\varphi_{d1}(e_1) \varphi_{d2}(e_2))/c. \tag{27}$$

Repeating this procedure iteratively, one has that

$$\phi_d(\mathbf{e}) \equiv \prod_{l=1}^{k} \varphi_{dl}(e_l) \tag{28}$$

for some continuous functions $\varphi_{dl}(.)$ defined on $[0,1]$. Substituting equation (28) into (18) yields the desired functional form.

Case 2. Assume $g_{\min} = 1$.

As is well-known,

$$\lim_{\theta \to 0} \left(\sum_{j=1}^{k} \left(g_j \phi_j(\mathbf{e})\right)^{\theta} \right)^{1/\theta} = \prod_{j=1}^{k} \left(g_j \phi_j(\mathbf{e})\right) \tag{29}$$

Defining $\mathbf{g} = (g_{\min}, \ldots, g_{\min}, g_d, g_{\min}, \ldots, g_{\min}) = (1, \ldots, 1, g_d, 1, \ldots, 1)$, equation (19) obtains. One can here essentially repeat the same steps as in Case 1 from equation (19) to (28) to obtain the desired functional form shown in equation (28). This completes the proof of the theorem.

Q.E.D.

Proof of Theorem 2.

Under the *Consistency condition*, it is trivial to prove that a correction function defined as $\varphi_{jl}(e_{il}) = 1+(\lambda_{jl}-1)e_{il}^{\alpha}$ with $\lambda_{jl} \in (0,1], \alpha > 0$ generates a multidimensional poverty index P that satisfies *Monotonicity on Excess Gaps* and *Uniform Scale Invariance*. Therefore, we only need to prove the reverse implication.

Consider a hypothetical scenario where $(G, E) \in \mathcal{G}_S^{n\times k} \times \mathcal{E}_S^{n\times k}$ and where all individuals are deprived in exactly the same attribute and non-deprived in the other ones. Since equation (3) and the *Consistency condition* are assumed to hold, it is tedious but not difficult to show that *Uniform Scale Invariance* can be written in terms of the correction functions $\varphi_{jl}(.)$ in the following way:

$$\varphi_{jl}(tx) - \varphi_{jl}(ty) = h(\varphi_{jl}(x) - \varphi_{jl}(y), t) \tag{30}$$

for each $\varphi_{jl}(.)$, for all $x, y \in [0,1]$ and for every $t > 0$ such that $tx, ty \in [0,1]$, where h is some function. Let $u = \varphi_{jl}(x) - \varphi_{jl}(y), v = \varphi_{jl}(y) - \varphi_{jl}(z)$. Then $u + v = \varphi_{jl}(x) - \varphi_{jl}(z)$. From (30) we have that

$$h(u+v,t) = \varphi_{jl}(tx) - \varphi_{jl}(tz) = (\varphi_{jl}(tx) - \varphi_{jl}(ty)) + (\varphi_{jl}(ty) - \varphi_{jl}(tz)) = h(u,t) + h(v,t) \tag{31}$$

By *Normalization* and *Monotonicity*, there exist parameters λ_{jl} such that $\varphi_{jl}(a) \in [\lambda_{jl}, 1] \subset (0,1]$ for all $a \in [0,1]$, so one must have that $u, v \in [\lambda_{jl} - 1, 1 - \lambda_{jl}] \subset (-1,1)$. Hence, from (31) we see that h satisfies the Cauchy equation with respect to the first argument whenever this one is included in the interval $[\lambda_{jl} - 1, 1 - \lambda_{jl}]$. It is now possible to extend this functional relationship to the set of real numbers $\mathbb{R}$. Take any $c \in \mathbb{R}$. It is always possible to write $c = p/q$ for some $p \in [\lambda_{jl} - 1, 1 - \lambda_{jl}], q \in [\lambda_{jl} - 1, 0) \cup (0, 1 - \lambda_{jl}]$. Define $H(c,t) = h(p,t)/(h(q,t)$. Using the same arguments as in Theorem 1 (Step 2), it is straightforward to check that $H(c,t)$ is well defined and that $H(c_1 + c_2, t) = H(c_1, t) + H(c_2, t)$ for any $c_1, c_2 \in \mathbb{R}$. By Continuity, h and H must be continuous in at least one point, so we can apply the characterization result found in Aczél (1966:34), according to which H can be written as $H(x,t) = cx\xi(t)$ for every $x \in \mathbb{R}$ for some continuous function $\xi(t)$ and for some constant c. Hence, since H is an extension of h, it is possible to rewrite (30) as

$$\varphi_{jl}(tx) - \varphi_{jl}(ty) = \psi(t)(\varphi_{jl}(x) - \varphi_{jl}(y)) \tag{32}$$

for some continuous function ψ. Again, this equation is restricted to the values of $x, y, tx, ty \in [0,1]$, but we will need to extend it to the set of positive numbers $\mathbb{R}_+$. As before, pick any $c \in \mathbb{R}_+$. It is always possible to write $c = p/q$ for some $p \in [0,1], q \in (0,1]$. Define

$$\Phi(c) := \frac{\varphi_{jl}(p) - \varphi_{jl}(0)}{\varphi_{jl}(q) - \varphi_{jl}(0)}$$

This function is well defined because if one has that $c_1 = c_2 \in \mathbb{R}_+$, then it is possible to write $c_1 = p_1/q_1$ and $c_2 = (mp_1)/(mq_1)$ for some $p_1 \in [0,1], q_1, m \in (0,1]$, so that, by equation (32)

$$\Phi(c_2) := \frac{\varphi_{jl}(mp_1) - \varphi_{jl}(0)}{\varphi_{jl}(mq_1) - \varphi_{jl}(0)} = \frac{\psi(m)(\varphi_{jl}(p_1) - \varphi_{jl}(0))}{\psi(m)(\varphi_{jl}(q_1) - \varphi_{jl}(0))} = \Phi(c_1)$$

Given that $\varphi_{jl}(.)$ satisfies Uniform scale invariance, one has that, if $\varphi_{jl}(x) - \varphi_{jl}(y) = \varphi_{jl}(x') - \varphi_{jl}(y')$, then $\varphi_{jl}(tx) - \varphi_{jl}(ty) = h(\varphi_{jl}(x) - \varphi_{jl}(y), t) = h(\varphi_{jl}(x') - \varphi_{jl}(y'), t) = \varphi_{jl}(tx') - \varphi_{jl}(ty')$ for any $x, y, x', y' \in [0,1]$. We will now check that Φ also satisfies that relationship. Assume that $\Phi(x) - \Phi(y) = \Phi(x') - \Phi(y')$ for any $x, y, x', y' \geq 0$. If one writes $x = p_1/q_1, y = p_2/q_2, x' = p_3/q_3, y' = p_4/q_4$, with $p_i \in [0,1], q_i \in (0,1]$ this means that

$$\frac{\varphi_{jl}(p_1) - \varphi_{jl}(0)}{\varphi_{jl}(q_1) - \varphi_{jl}(0)} - \frac{\varphi_{jl}(p_2) - \varphi_{jl}(0)}{\varphi_{jl}(q_2) - \varphi_{jl}(0)} = \frac{\varphi_{jl}(p_3) - \varphi_{jl}(0)}{\varphi_{jl}(q_3) - \varphi_{jl}(0)} - \frac{\varphi_{jl}(p_4) - \varphi_{jl}(0)}{\varphi_{jl}(q_4) - \varphi_{jl}(0)} \tag{33}$$

Now, for any $t > 0$, by equation (32) one has that

$$\Phi(tx) - \Phi(ty) = \frac{\varphi_{jl}(tp_1) - \varphi_{jl}(0)}{\varphi_{jl}(q_1) - \varphi_{jl}(0)} - \frac{\varphi_{jl}(tp_2) - \varphi_{jl}(0)}{\varphi_{jl}(q_2) - \varphi_{jl}(0)} = \psi(t)\left(\frac{\varphi_{jl}(p_1) - \varphi_{jl}(0)}{\varphi_{jl}(q_1) - \varphi_{jl}(0)} - \frac{\varphi_{jl}(p_2) - \varphi_{jl}(0)}{\varphi_{jl}(q_2) - \varphi_{jl}(0)}\right) \tag{34}$$

by equations (32) and (33) the last expression is equal to

$$\psi(t)\left(\frac{\varphi_{jl}(p_3) - \varphi_{jl}(0)}{\varphi_{jl}(q_3) - \varphi_{jl}(0)} - \frac{\varphi_{jl}(p_4) - \varphi_{jl}(0)}{\varphi_{jl}(q_4) - \varphi_{jl}(0)}\right) = \frac{\varphi_{jl}(tp_3) - \varphi_{jl}(0)}{\varphi_{jl}(q_3) - \varphi_{jl}(0)} - \frac{\varphi_{jl}(tp_4) - \varphi_{jl}(0)}{\varphi_{jl}(q_4) - \varphi_{jl}(0)} = \Phi(tx') - \Phi(ty') \tag{35}$$

To sum up, using equations (34) and (35) we have checked that Φ satisfies the following: For any $x, y, x', y' \geq 0$, $\Phi(x) - \Phi(y) = \Phi(x') - \Phi(y')$ implies $\Phi(tx) - \Phi(ty) = \Phi(tx') - \Phi(ty')$ for any $t > 0$. Given the fact that the continuity of $\varphi_{jl}(.)$ ensures that Φ must be continuous everywhere, we can apply the result of Eichhorn and Gleissner (1988:24-26) and Aczél (1988:6) according to which we must have either $\Phi(c) = \theta c^{\alpha} + \gamma$ (for all $c \geq 0$ and for some constants θ, α, γ) or $\Phi(c) = \theta \log c + \gamma$ (for all $c > 0$ and for some constants θ, γ). Given the fact that $\Phi(0) = \Phi(0/q) = (\varphi_{jl}(0) - \varphi_{jl}(0))/(\varphi_{jl}(q) - \varphi_{jl}(0)) = 0$, one must have that $\Phi(c) = \theta c^{\alpha}$ for some constants $\theta \in \mathbb{R}, \alpha > 0$. Moreover, one has that $\Phi(1) = \Phi(p/p) = (\varphi_{jl}(p) - \varphi_{jl}(0))/(\varphi_{jl}(p) - \varphi_{jl}(0)) = 1$, so one must have that $\Phi(c) = c^{\alpha}$ for some constant $\alpha > 0$. Hence, when $c \in [0,1], \Phi(c) = \Phi(c/1) = (\varphi_{jl}(c) - \varphi_{jl}(0))/(\varphi_{jl}(1) - \varphi_{jl}(0)) = c^{\alpha}$. This means that the correction function can be written as

$$\varphi_{jl}(c) = (\varphi_{jl}(1) - \varphi_{jl}(0))c^{\alpha} + \varphi_{jl}(0) \tag{36}$$

By *Normalization*, $\varphi_{jl}(0) = 1$, so we have obtained the desired functional form $\varphi_{jl}(c) = (\varphi_{jl}(1) - 1)c^{\alpha} + 1$ for some $\alpha > 0$. Defining $\lambda_{jl} := \varphi_{jl}(1)$ we are done.

Q.E.D.

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