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Abstract

In this paper we present a fuzzy approach for the measurement of multidimensional poverty in European countries from the perspective of the 'leaving no one behind' principle underlying the Sustainable Development Goals (SDGs). In particular, we compute the degree to which an individual is 'left behind' in a specific dimension of poverty. Afterwards, we propose alternatives to measure the extent to which an individual is 'left behind' in a multidimensional setting. We illustrate our proposal taking as a reference the 'at risk of poverty or social exclusion' (AROPE) framework (dimensions, indicators and union criteria). The results highlight that although the average 'left behind' (LB) level and AROPE are largely consistent across countries, our fuzzy measure provides valuable additional information, both in individual and aggregate terms, on the individuals who are further away from those better positioned and how far they are from them.

Keywords: Fuzzy sets, multidimensional poverty, AROPE, 'left behind'

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1. Introduction

In 2015, world leaders adopted 17 Global Goals – the Sustainable Development Goals (SDGs) – that have the potential to end poverty, reduce inequality and tackle climate change in 15 years, among other challenges. One of the pillars of the 2030 Agenda for Sustainable Development and the SDGs, and one which represents a critical improvement over the Millennium Development Goals (MDGs), is the pledge to 'leave no one behind'. In committing to the realization of the 2030 Agenda for Sustainable Development, Member States recognized that the dignity of the individual is fundamental and that the Agenda's goals and targets should be met for all nations and people and for all segments of society. Furthermore, they endeavored to reach first those who are furthest behind.

This way, the Sustainable Development Agenda has brought inequalities to center stage. Numerous goals and targets include a focus on inequalities and the advancement of communities that have historically experienced discrimination. Explicitly, goal 10 focuses on reducing inequalities within and among countries, even though this goal is undermined and distorted by the targets and indicators, which set an agenda for inclusion rather than for reducing inequalities. Behind it lies the idea that a focus on absolute goals obviates the need for relative indicators of inequality. For example, if countries eliminate avoidable child deaths or achieve universal education, it follows that nobody has been left behind for that indicator. The problem is, however, that growing inequality is compatible with overall progress in the transition towards the absolute goal.

It should also be remembered that the revealing test for whether the SDGs will truly 'leave no one behind' is not whether the SDG goals and targets include such (aspirational) language, but whether this language will translate into the implementation of the goals on the basis of equality and non-discrimination. In that regard, monitoring will play an important role. As metrics pegged to specific targets, indicators have the power to concentrate effort and attention. Moving beyond aggregate outcomes will require that the data related to these indicators be disaggregated along lines able to meaningfully demonstrate the existence, magnitude and interplay of multiple forms of inequalities. Moreover, to leave no one behind, it is not enough to address the problems of those at the bottom; we need to analyze inequality from an individual viewpoint. Thus, the starting point requires a precise understanding and identification of those who are left behind and to what extent.

This is precisely the issue this papers attempts to address. We use the fuzzy approach to measure the degree to which an individual is left behind in a specific dimension, as well as alternatives to measure the degree an individual is 'left behind' from a multidimensional perspective. To this end, we follow Temkin (1993) who proposed that inequality can be viewed in terms of complaints of individuals located at disadvantaged positions in the dimension under study. The society's highest values of the dimension in question are the reference point for all, and everybody except the best positioned individual has a legitimate complaint. Consequently, a measure of the degree an individual is left behind is the sum of his/her shortfalls from all persons better positioned than him/her in the dimension. This is a novel proposal that attempts to measure for the first time the 'left behind' principle, allowing us to identify those left behind and quantify how much they are left behind.

This procedure can be applied to any dimension. We present our proposal by illustrating the'left behind' principle in terms of multidimensional poverty, specifically in relation to the concept of 'at risk of poverty or social exclusion' (AROPE). This is another significant novelty of our investigation. To the best of our knowledge, the fuzzy sets applied in our analysis are original in the fuzzy approach literature regarding poverty. In this regard, as an indicator of multidimensional poverty, AROPE contains three dimensions, (two continuous dimensions and discrete), that allow us to illustrate the distinctivefeaures of our proposal in both types of dimensions, and to propose alternatives to the joint measurement of dimensions. We illustrate our proposal with the measurement of 'left behind' in multidimensional poverty for 26 European countries (EU-28 countries except Bulgaria, Croatia, Malta and Romania, plus Iceland and Norway) in the years 2006 and 2016 using European Statistics on Income and Living Conditions (EU–SILC) microdata. The empirical illustration highlights the significant advantages of the fuzzy measurement of the 'leaving no one behind' principle in terms of multidimensional poverty.

The paper is structured as follows AROPE and the basic concepts of fuzzy sets are introduced in the next section. The methodology and our contribution are then described in section 3. The results of the empirical illustration are presented and discussed in section 4. Finally, section 5 concludes.

2. AROPE and fuzzy multidimensional poverty

In the conventional approach, poverty is characterized by a dichotomization of the population into the poor and the non-poor, which are defined in relation to some chosen poverty line. Poverty may be defined, for instance, as a certain percentage (50, 60 or 70%) of the mean or the median of the equivalent income or consumption distribution. However, poverty is a complex phenomenon that cannot be reduced solely to the monetary dimension and must also take account of non-monetary indicators of living conditions. In order to capture the multidimensional nature of poverty, in 2010 the AROPE indicator was established. Since then, it constitutes the pivotal indicator of living conditions and poverty in the EU. This multidimensional indicator was thought to be more appropriate than just considering the monetary indicator based solely on relative income. AROPE combines three facets already computed by the EU before 2010: at risk of poverty, material deprivation and low work intensity of households. The first is the standard EU at-risk-of-poverty rate, which identifies if an individual has an equivalized[2] disposable income below the at-risk-of-poverty threshold, set at 60% of the national median equivalized disposable income. The second is severe material deprivation and refers to a state of economic strain and durables, defined as the enforced inability to pay unexpected expenses; afford a one week annual holiday away from home; a meal involving meat, chicken or fish every second day; the adequate heating of a dwelling; durable goods like a washing machine, color television, telephone or car; being confronted with payment arrears (mortgage or rent, utility bills, hire purchase installments or other loan payments). In particular, the severe material deprivation variable expresses the inability to afford at least four of nine items considered by most people to be desirable or even necessary to lead an adequate life. The third variable, low work intensity, identifies if a person is living in a household where the members of working age worked less than 20% of their total potential during the income reference period. A working-age person is a person aged 18-59 years, with the exclusion of students in the age group between 18 and 24 years.

Combining these three distinct dimensions, an individual is considered to be at risk of poverty or social exclusion if he or she is at risk of poverty,

[2]EUROSTAT recommends the use of the OECD modified equivalence scale to compute equivalent units. This scale attaches a value of 1 to the first adult in the household, 0.5 to each remaining adult, and 0.3 to each member younger than 14.

or is severely materially deprived or lives in a household with low work intensity. In other words, the indicators are used to identify the target group so that meeting any of the three criteria suffices for an individual to be included among those counted as poor or socially excluded.

Nevertheless, it is obvious that in reality multidimensional poverty is not an attribute that characterizes an individual in terms of its presence or absence (see, for example, Zedini & Belhadj (2016)), but is rather a predicate that manifests itself in varying shades and degrees (ambiguity). Thus, the fuzzy approach allows us to consider multidimensional poverty as a matter of degree rather than an attribute that is simply present or absent for individuals in the population. Moreover, once we consider that multidimensional poverty is not a dichotomous attribute, we can identify those who are further away from the better positioned individuals and how far they are. We make use of the fuzzy approach, therefore, to measure the degree of multidimensional poverty of each individual and to gauge how far they left behind.

Let us recall, in line with Zadeh (1965), that one of the main ideas of the fuzzy systems theory is that, in addition to 'fully belonging' (truth value 1) and 'fully not belonging' (truth value 0), there can also be other cases (other truth values) of an element belonging to a fuzzy set; hence being a member of a fuzzy set is a graded property. This is contrary to the principle of bivalence which states that a property either applies to an element or does not.

A number of results can be found in the literature concerning fuzzy multidimensional poverty. Some authors have introduced alternative approaches guided by the duality of poverty (see, for instance, Garcia & Machado (1998); Chiappero Martinetti (2000); Lelli (2001)). Additionally, Bourguignon & Chakravarty (2003) and Chakravarty (2016, chapter 4) have contributed to the development of an axiomatic methodology with diverse fuzzy set approaches to the measurement of multidimensional poverty.

Apart from the fuzzy axiomatic approaches, other authors, far from the works previously mentioned, have introduced the fuzzy approach in different ways. For example, Cerioli & Zani (1990) proposed a fuzzy approach for the monetary dimension of poverty with the definition of the fuzzy set where the membership function is defined as a transition zone between two states of poverty; a zone over which there was a very simple linear function of the monetary variable between zero and one.

Later, Betti et al. (2006) propose a membership function that combines their previous membership functions (see Cheli & Lemmi (1995), Cheli & Betti (1999), Betti & Verma (1999), Cheli & Betti (2004)). They called this indicator the Fuzzy Monetary Incidence, which takes into account: $(1 - F(x_i))$ membership function, which represents the proportion of individuals with income greater than the income of individual i, where $F(x_i)$ is the distribution function; and $(1 - L(F(x_i)))$ is the membership function, that represents the income share of those with an income greater than the income of individual i, where $L(F(x_i))$ is the the value of the Lorenz curve for the individual i. They define this measure as: $\mu : U \to [0, 1]$,

$$\mu(i) = (1 - F(x_i))^{\alpha}(1 - L(F(x_i))) \quad \text{for all} \quad i \in U, \tag{1}$$

where U is the population set, x_i is the equivalent income of individual i and α is a parameter.[3]

In this paper we propose a definition of fuzzy sets that captures the shortfall of individuals with respect to those with better position, which is based on the widely known concept of relative deprivation of an individual i introduced by Yitzhaki (1979) and Hey & Lambert (1980).

Hey & Lambert (1980) define the relative deprivation of individual i with income x_i with respect to another individual j with income x_j as the following linear function

$$P(x_i, x_j) = \begin{cases} x_j - x_i & \text{if } x_j \geq x_i \\ 0 & \text{if } x_j \leq x_i \end{cases} \tag{2}$$

Using the above concept, these authors define the mean deprivation of an individual i with income x_i as follows:

Definition 1. Let U be the population set, η be the average income of U and F be the income distribution function. The *mean deprivation of individual i with income* x_i is defined as follows:

$$P(x_i) = \int_{-\infty}^{+\infty} P(x_i, x_j)\, dF(x_j) = \eta(1 - L(F(x_i))) - x_i(1 - F(x_i)) \tag{3}$$

where $P(x_i, x_j)$ is defined by Equation (2) for all $i, j \in U$ and $L(F(x_i))$ is the value of the Lorenz curve for income x_i.

[3]The arbitrary parameter α may be chosen, for example, so that the mean of $\mu(i)$ equals the head count ratio H. For a detailed review of the definition of functions and parameter α, see, for instance, Cheli & Betti (1999), Betti & Verma (1999), Cheli & Betti (2004), Betti et al. (2006) and Betti (2008).

It is worth noting that the fuzzy sets defined in our analysis combine the information contained in the distribution function and the Lorenz curve in a way that has meaning in itself. It measures the 'left behind' notion as it is the sum of the shortfalls of each individual with respect to those who are better positioned. Moreover, the average of such individuals' shortfalls is the well-known Gini index[4] of inequality.

3. Methodology

3.1. Preliminary definitions

As an underlying structure for considering the generalization to a fuzzy framework, we will consider a complete residuated lattice $\mathbb{L} = (L, \leq, \otimes, \rightarrow, 0, 1)$, that is an algebra where $(L, \leq, 0, 1)$ is complete lattice, the least element is 0 and the greatest element is 1, $(L, \otimes, 1)$ is a commutative monoid and $(\otimes, \rightarrow)$ is an adjoin couple ($x \leq y \rightarrow z \quad \text{iff} \quad x \otimes y \leq z$) for all $x, y, z \in L$ ($\leq$ denotes the lattice ordering).[5] We will denote the supremum and infimum operation in the lattice with the symbols $\vee$ and $\wedge$, respectively.

An $\mathbb{L}$-fuzzy set on U is a mapping $A : U \rightarrow L$ where $A(u)$ is called the degree of membership of u in A.

The set of all $\mathbb{L}$-fuzzy[6] sets on U is denoted by L^U. Let $A, B \in L^U$:

i) A is said to be included in B, denoted as $A \subseteq B$ if $A(u) \leq B(u)$ for all $u \in U$.

ii) The union (resp. intersection) of A and B is defined as the fuzzy set $(A \cup B)(u) = A(u) \vee B(u)$ (resp.$(A \cap B)(u) = A(u) \wedge B(u)$) for all $u \in U$.

ii) The multiplication (resp. implication) of A and B is defined as the fuzzy set $(A \otimes B)(u) = A(u) \otimes B(u)$ (resp. $(A \rightarrow B)(u) = A(u) \rightarrow B(u)$) for all $u \in U$.

For $n \in \mathbb{N}$, we denote the *n-power of* $a \in L$ *with respect to* $\otimes$ by $a^{\otimes n}$ i.e. $a^{\otimes n} = a \otimes a ... \otimes a$ (n-times). The formal definition is as follows,

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[4] $G = \frac{\sum_i^N \sum_j^N |x_i - x_j|}{2N^2\eta}$, where N is the size of the population set, η is the average of the variable X, and $x_i, x_j \in X$ are the value of the variable for individuals i and j, respectively.

[5] For more details, see, for example, Birkhoff (1967), Bělohlávek (2002) and Davey & Priestley (2002).

[6] From now on, when no confusion arises, we will omit the prefix $\mathbb{L}$.

Definition 2. Let $\mathbb{L} = (L, \leq, \otimes, \rightarrow, 0, 1)$ be a residuated lattice. For a nonnegative integer n, the *n-power* of $a \in L$ is defined by $a^0 = 1$ and $a^{\otimes n+1} = a^{\otimes n} \otimes a$.

Let us now turn out attention to the residuated lattices defined on $L = [0,1]$ and recall the following concepts.

Definition 3. A *t-norma* is a binary operation on $[0,1]$ which is associative, commutative, monotone, and with 1 acting as its unit element, i.e., $\otimes$ is a mapping $\otimes : [0,1] \times [0,1] \rightarrow [0,1]$ satisfying $(x \otimes y) \otimes z = x \otimes (y \otimes z)$, $x \otimes y = y \otimes x$, $y_1 \leq y_2$ implies $x \otimes y_1 \leq x \otimes y_2$ and $x \otimes 1 = x$.

The following technical lemma shows that it is possible to define r-power of $a \in [0,1]$ with respect to $\otimes$ for any positive rational number r.

Lemma 1. *For a t-norma that verifies elements* 0 *and* 1 *as its only idempotents (i.e., $a \otimes a < a$ for any $0 \neq a \neq 1$), we have:*

i) *if $0 < a^{\otimes n} < 1$ and $n < m$ then $a^{\otimes m} < a^{\otimes n}$ for all $n, m \in \mathbb{N}$;*

ii) *for each positive $a < 1$ and each $n \in \mathbb{N}$ there is a unique b such that $b^{\otimes n} = a$.*

The above lemma (item ii)) justifies the following definition.

Definition 4. For $a > 0$, $a^{\otimes \frac{1}{n}}$ is the unique b such that $b^{\otimes n} = a$; $0^{\otimes \frac{1}{n}} = 0$. For positive integers m, n and rational $r = \frac{m}{n}$, we define $a^{\otimes r} = (a^{\otimes \frac{1}{n}})^{\otimes m}$.

3.2. Fuzzy dimensional measure

In this subsection, we describe the fuzzy methodology for the construction of measures of 'left behind' in terms of multidimensional poverty. We consider two step procedure. We first compute the degree to which the individual is 'left behind' in a specific dimension of poverty, then aggregate across dimensions for each individual and finally aggregate across individuals. The analysis of multidimensional poverty involves diverse information from very different kinds of data. In particular, the main purpose of this subsection is to define fuzzy sets for a given dimension (whether continuous or non-continuous). First, we focus on the definition of fuzzy set for a continuous dimension. The idea underlying the construction of the measure is a transformation of the concept of the mean relative deprivation for individual i introduced in Definition 1.

Definition 5. Let $\mathbb{L} = (L, \leq, \otimes, \rightarrow, 0, 1)$ be a complete residuated lattice and h be a continuous dimension. Given a population set U, for each individual $i \in U$, the *fuzzy set* f_h is defined as the mapping $f_h : U \rightarrow L$ where

$$f_h(i) = \frac{P(x_{h,i})}{\eta_h}$$

, where $x_{h,i}$ is the value of the continuous dimension h for individual i and η_h is the average of x_h.

Thus, for h being income (work intensity), $f_h(i)$ represents the *degree the individual $i \in U$ is 'left behind' in terms of monetary poverty (work intensity).*

Remark 1. Note that in this paper we consider that the value of the continuous dimension h for individual i is non-negative.

Proposition 1. *Let U be a population set, $\mathbb{L}$ be a complete residuated lattice and h be a continuous dimension. Then, the map $f_h : U \rightarrow L$, where $f_h(i) = \frac{P(x_{h,i})}{\eta_h}$ verifies that $0 \leq f_h(i) \leq 1$ for all $i \in U$.*

PROOF. $P(x_{h,i})$ defined in Definition 1. Consider $i \in U$ and let us prove that

$$0 \leq (1 - L(F(x_{h,i})) - \frac{x_{h,i}}{\eta_h}(1 - F(x_{h,i})) \leq 1,$$

By definition of F and L, we have that,

$$0 \leq (1 - F(x_{h,i})) \leq 1, \tag{4}$$

$$0 \leq (1 - L(F(x_{h,i}))) \leq 1, \tag{5}$$

$$(1 - F(x_{h,i})) \leq (1 - L(F(x_{h,i}))). \tag{6}$$

Using equations (4) and (5), and the fact that $\frac{x_{h,i}}{\eta_h} \geq 0$ for all $i \in U$, we obtain that $(1 - L(F(x_{h,i})) - \frac{x_{h,i}}{\eta_h}(1 - F(x_{h,i}))) \leq 1 - L(F(x_{h,i})) \leq 1$ for all $i \in U$. On the other hand, it is straightforward that $P(x_{h,i}) \geq 0$, by Definition 1 and by Equation (2). □

Proposition 2. *Let U be a population set, $\mathbb{L}$ complete residuated lattice and h a continuous dimension. Consider the mapping $f_h : U \rightarrow L$ such that $f_h(i) = \frac{P(x_{h,i})}{\eta_h}$, if $x_{h,i} \leq x_{h,j}$ then $f_h(j) \leq f_h(i)$ for all $i, j \in U$.*

PROOF. Consider $i, j \in U$ such that $x_{h,i} \leq x_{h,j}$ and let us prove that $\frac{P(x_{h,j})}{\eta_h} \leq \frac{P(x_{h,i})}{\eta_h}$.

By definition of F and Remark 1, we have

$$-\frac{x_{h,j}}{\eta_h}(1 - F(x_{h,j})) \leq -\frac{x_{h,i}}{\eta_h}(1 - F(x_{h,j})), \tag{7}$$

and

$$-\frac{x_{h,i}}{\eta_h}(1 - F(x_{h,i})) \leq -\frac{x_{h,i}}{\eta_h}(1 - F(x_{h,j})). \tag{8}$$

Now, subtracting equations (7) and (8) we obtain

$$-\frac{x_{h,j}}{\eta_h}(1 - F(x_{h,j})) \leq -\frac{x_{h,i}}{\eta_h}(1 - F(x_{h,i}) \tag{9}$$

Therefore, using first equation (9) and second the fact that $1 - L(F(x_{h,j})) \leq 1 - L(F(x_{h,i}))$, we have

$$\begin{gathered}(1 - L(F(x_{h,j}))) - \frac{x_{h,j}}{\eta_h}(1 - F(x_{h,j})) \leq \\ (1 - L(F(x_{h,j}))) - \frac{x_{h,i}}{\eta_h}(1 - F(x_{h,i})) \leq \\ (1 - L(F(x_{h,i}))) - \frac{x_{h,i}}{\eta_h}(1 - F(x_{h,i}).\end{gathered}$$

Therefore, $f_h(j) \leq f_h(i)$ for all $i, j \in U$. □

Example 1. We consider that the population set $U = \{1, 2, 3\}$ and h is a continuous dimension, specifically the income of individuals in the population, for a country C, as described in the following table:

Individual	$x_{h,i}$	$f_h(i)$
$i = 1$	300	0.2
$i = 2$	500	0
$i = 3$	200	0.4

Therefore, the fuzzy set above is described as for the individuals $i = 1$, $i = 2$ and $i = 3$ are left behind in monetary poverty with degrees of 0.2, 0 and 0.4, respectively.

The second part of this section is devoted to considering the case of a non-continuous dimension such as material deprivation. This variable can be described by categories with different symptoms. These categories indicating material deprivation often take the form of simple ‘yes/no’ dichotomies (such as the presence or absence of enforced lack of certain goods or facilities), and sometimes ordered polytomies. Therefore, the starting point for studying non-continuous dimensions in deprivation analysis is the selection of different deprivation categories ($j \in \{1, 2, ...k\}$) with their symptoms. A numerical value (rank) is assigned to each deprivation symptom ($c \in \{1, 2, ..., n\}$) where the deprivation symptoms are arranged from the most deprived ($c = 1$) to the least deprived ($c = n$) situation.[7]

Considering the previous statements, we focus on the transformation of non-continuous dimensions into continuous dimensions, since we intend to use the definition of fuzzy set given in Definition 5 for any type of dimension. The crux of the transformation is related to the definition of a number (a score) which collects all the information of the different deprivation categories with their different deprivation symptoms. The steps for any non-continuous dimension h are as follows:

1. For each deprivation category we determine a deprivation score that is defined as follows:

   Definition 6 (Cheli & Lemmi (1995)). Let U be a population set and h a non-continuous dimension with deprivation categories $j \in \{1 \ldots k_h\}$. Consider the ordered set of deprivation symptoms of each category j, $c_{h,j} \in \{1, 2, \ldots, n_j\}$, where the most deprived is $c_{h,j} = 1$ and the least deprived is $c_{h,j} = n_j$. The *deprivation score in category j for individual i in the non-continuous dimension h* is defined as follows

   $$e_{h,j,i} = \frac{1 - F(c_{h,j,i})}{1 - F(1)} \tag{10}$$

   where $F(c_{h,j,i})$ is the value of the j-th category distribution function for the i-th individual in the dimension h.

[7]For example, we study two deprivation categories ($j = 1, 2$). The first category ‘to have a car’ ($j = 1$), has two deprivation symptoms, one for not having a car ($c = 1$) and another for having a car ($c = 2$). The second category, ‘defaults home mortgage loan’ ($j = 2$), has three deprivation symptoms: one for defaults for more than sixth months ($c = 1$), another one for defaults for one month ($c = 2$) and the last one for no default ($c = 3$).

We use this transformation as we cannot calculate the shortfalls with respect to 'better off' individuals because the value of the symptom has no meaning in itself. What we capture is the proportion of individuals that are 'better off' than individual i.
Note that the above formulation for $e_{h,j,i}$ is identical for the most common case, the dichotomous category, $e_{h,j,i} = 1$ (deprived) or $e_{h,j,i} = 0$ (non-deprived).

2. We collect the information in an overall deprivation score that indicates the deprivation of each individual of the population in the specific non-continuous dimension. For this purpose we define a weighting system in the dimension that assigns a specific weight to each category j. Before stating the following definitions, let us introduce the notation as follows:

Notation 1. *Let $X_1, ..., X_n$ be n variables. We will use the following notation: for all $i \in \{1, ..., n\}$*

$$R^2_{X_i,\ X_{-i}} = R^2_{X_i, X_1, ..., X_{i-1}, X_{i+1}, ..., X_n},$$

where $R^2_{X_i, X_1, ..., X_{i-1}, X_{i+1}, ..., X_n}$ is the well-known coefficient of determination for a multiple linear regression model in which X_i is the dependent variable and $X_1, ..., X_{i-1}, X_{i+1}, ..., X_n$ are the independent variables.

In order to combine information from different categories into one measure of individual deprivation, we compute weighted averages of the $e_{h,j,i}$ for each individual. The weights of each category should have an inverse relation to the deprivation score in the category (Deutsch & Silber (2005)), so that less frequent deprivation get higher weights. At the same time, categories that provide redundant information should be penalized. Taking these two premises in mind, we define:

Definition 7. Let U be a population set and h be a non-continuous dimension. We define the weight $w_{h,j} = w^a_{h,j} w^b_{h,j}$, where

- $w^a_{h,j} = 1 - \sum_{i=1}^{N} \frac{e_{h,j,i}}{N}$ for all $j = 1, 2, \ldots, k_h$ and for all $i \in U$.
- and $w^b_{h,j} = 1 - R^2_{e_{h,j,i},\ e_{h,-j,i}}$ for all $j = 1, 2, \ldots, k_h$ and for all $i \in U$.

$w^a_{h,j}$ attaches more weight to categories in which the proportion of individuals of the population with deprivation in category j is smaller. And, $w^b_{h,j}$ captures the proportion of the variance for a dependent variable (category j) that is not explained by independent variables (categories $1, \dots, j-1, j+1, \dots, k_h$ in a lineal multiple regression model). That is, we attach less weight to categories with redundant information.

3. Finally, using the above weighting system we define the deprivation score for each individual of the population.

   Definition 8. Let U be the universe set and h a non-continuous dimension with $j \in \{1, \dots, k_h\}$ deprivation categories. The *deprivation score for individual* i is given by

$$e_{h,i} = \frac{\displaystyle\sum_{j=1}^{k_h} w_{h,j} e_{h,j,i}}{\displaystyle\sum_{j=1}^{k_h} w_{h,j}}$$

   for all $i \in U$, $j \in \{1, 2, \dots, k_h\}$.

Note that in Definition 8, if $e_{h,i} = 1$, individual $i \in U$ has the maximum deprivation score in the dimension h and if $e_{h,i} = 0$, individual $i \in U$ has no-deprivation in dimension h.

In order to facilitate the understanding of the different notions and definitions introduced in the possible composition of a non-continuous dimension, we illustrate it by means of a toy example.

Example 2. Let U be a population set with three individuals, $i \in \{1, 2, 3\}$. Let h be the material deprivation dimension (non-continuous) with two deprivation categories ($j \in \{1, 2\}$). The category 'to have a car' (j=1), has two deprivation symptoms one of them for not having a car (c=1) and another one for having a car (c=2). The category 'defaults home mortgage loan' ($j = 2$), has three deprivation symptoms: one for defaults for more than sixth months ($c = 1$), another one for defaults for one month (c=2) and the last one for no default (c=3).

The deprivation scores in each of the different categories for each individual are as follows:

Individual	$c_{h,1,i}$	$c_{h,2,i}$	$e_{h,1,i}$	$e_{h,2,i}$
$i = 1$	1	2	1	0.5
$i = 2$	2	3	0	0
$i = 3$	1	1	1	1

The weighting system is the following products of $w^a_{h,j}$ and $w^b_{h,j}$ for each deprivation category $j \in \{1, 2\}$,

Deprivation category	$w^a_{h,j}$	$w^b_{h,j}$	$w_{h,j}$
$j = 1$	0.33	0.91	0.30
$j = 2$	0.5	0.91	0.45

Therefore, we calculate the deprivation score for each individual as follows:

Individual	$e_{h,i}$
$i = 1$	0.7
$i = 2$	0
$i = 3$	1

We can say that individual 3 has the maximun deprivation score, while 2 is not deprived from h and individual 1 has a deprivation score of 0.7.

Remark 2. Note that, from Proposition 2, for example, if there exist two individuals with different incomes then the individual with the smallest income has a larger degree of monetary poverty than the other individual. As expected, this property of these fuzzy sets is preserved for all continuous dimensions.

However, bear in mind that in a non-continuous dimension the greater $e_{h,i}$, the greater the deprivation score of the dimension ($f_h(i)$). In order to standardize with the continuous dimension, we transform the score $e_{h,i}$ as follows:

$$x_{h,i} = 1 - e_{h,i}.$$

Then, we would apply the fuzzy set defined in Definition 5 to $x_{h,i}$.

Let us finish this subsection with an example.

Example 3. For the material deprivation dimension h and the population U given in Example 2, the fuzzy set f_h (defined by Definition 5) that represents the *degree an individual is left behind in the dimension* h is as follows:

Individual	$e_{h,i}$	$x_{h,i}$	$f_h(i) = \frac{P(x_{h,i})}{\eta_h}$
$i = 1$	0.7	0.3	0.54
$i = 2$	0	1	0
$i = 3$	1	0	1

3.3. Fuzzy multidimensional measure

In the previous sections we have considered each dimension of the study of multidimensional poverty independently as a fuzzy set.

In this section, we tackle the next step of interest in multidimensional analysis and aggregate across dimensions for each individual, thus permitting an unambiguous ranking of individuals in the population.

Fuzzy set operations are a generalization of the corresponding crisp set operations in the sense that the former exactly reproduce the latter (see, for example, Section 3.1). This confirms that there is more than one way to formulate a composite indicator of multidimensional poverty with fuzzy sets.

The choice of alternative formulations has to be based primarily on substantive grounds: depending on the context and objectives of the application, some options are more appropriate than others.

The proposed measurement framework in this study is generic in the sense that it can be used to summarize performances in any policy setting. We follow the philosophy of AROPE to define the overall measure of the degree to which an individual is 'left-behind' in multidimensional poverty. We consider that given that the three dimensions of AROPE do indeed capture different categories of deprivation, we will use the union of fuzzy sets as an aggregation measure.

Definition 9. Let $h \in \{1...H\}$ be dimensions, U the population set and $([0,1], \leq, \otimes, \rightarrow, 0, 1)$ the residuate lattice. Let f_h be the fuzzy set of dimension h.

The mapping $\alpha : U \rightarrow [0,1]$ is called *the degree an individual is 'left behind' regarding the dimension* $h \in \{1...H\}$ and is defined as follows:

$$\alpha(i) = (\bigcup_{h=1}^{H} f_h)(i) = \bigvee_{h=1}^{H} f_h(i)$$

for all $i \in U$.

An individual $i \in U$ is totally 'left behind' in multidimensional poverty if $\alpha(i) = 1$ and he/she is at the bottom of the ranking. Individual $i \in U$ is not 'left behind' at all if $\alpha(i) = 0$, that is, he/she leads the ranking.

Let us think of a specific situation in which dimensions can be weighted differently depending on the country context. In this case, the definition of the fuzzy set which measures the degree an individual is 'left behind' regarding the multidimensional analysis is as follows:

$$\alpha^*(i) = (\bigcup_{h=1}^{H} f_h^{\otimes \beta_{h,C}})(i) = \bigvee_{h=1}^{H} (f_h(i)^{\otimes \beta_{h,C}})$$

for all $i \in U$, where $\beta_{h,C}$ is the weight of the dimension h in country C.

Note that it is straightforward that $0 \leq \alpha^*(i) \leq 1$ for all $i \in U$.

An example is worked out below to illustrate the previous situation.

Example 4. We consider the population set $U = \{1, 2, 3\}$ and study the degree an individual is 'left behind' in terms of multidimensional poverty. In this case, we consider two dimensions: dimension $h = 1$ defined in Example 1 and dimension $h = 2$ defined in Example 2, specifically,

Individual	$f_1(i)$	$f_2(i)$	$\alpha(i)$
$i = 1$	0.54	0.2	0.54
$i = 2$	0	0	0
$i = 3$	1	0.4	1

We can say that individual 3 with degree 1 is the most 'left behind' in terms of multidimensional poverty, that is, he is the worst off, while individual 2 is not 'left behind' at all and is the best off, and individual 1 is 'left behind' in terms of multidimensional poverty with degree 0.54.

4. Empirical analysis

This section reports the empirical results of the average 'left behind' (LB) level in terms of multidimensional poverty for 26 European countries based on the [dataset]EU SILC (2006) and [dataset]EU SILC (2016) rounds[8] of the EU-SILC data set. This data set includes timely and com-

[8]We analyze data on 132, 323 households in 2006 and 130, 125 households in 2016. Households composed only of children, of students aged less than 25 and/or people aged 60 or more are completely excluded from the indicator calculation. Additionally, we eliminate households from the sample that did not provide information on one or more of the dimensions. Table $A1$ in the appendix shows the number of observations by country and year.

parable cross-sectional and longitudinal multidimensional microdata on income, poverty, social exclusion and living conditions. It is at curretly the main source of information on living standards in the EU and is based on a common framework with a common set of target variable definitions and rules. The years have been chosen to highlight changes over a decade starting before and ending after the period of economic crisis.

Let us recall that the multidimensional poverty indicator AROPE includes three facets: at risk of poverty, severe material deprivation and low work intensity of households. Combining these three distinct dimensions, an individual is considered to be at risk of poverty or social exclusion if he or she is at risk of poverty, or is severely materially deprived or lives in a household with very low work intensity. Thus, the measure only checks whether a person satisfies achievement in the dimension or not. From this multidimensional poverty perspective, an individual may be at risk of poverty or social exclusion in one or more dimensions, not taking into consideration the number of dimensions and degrees in which the person falls short. As explained above, we propose to complement and extend the information provided by the AROPE rate by using a fuzzy approach that allows us to identify those who are further away from the better positioned individuals and how far they are. Table 1 reports the results of the LB and the AROPE rate considering the three dimensions of poverty used by EUROSTAT for the years 2006 and 2016.

Table 1. LB and AROPE rate for 2006 and 2016 by country.

	2006				2016			
Country	LB	rank	AROPE	rank	LB	rank	AROPE	rank
PL	0.4436	(1)	0.4098	(1)	0.3800	(10)	0.2467	(8)
LV	0.4217	(2)	0.3786	(2)	0.3852	(8)	0.2398	(10)
HU	0.4202	(3)	0.3261	(4)	0.3667	(14)	0.2850	(4)
PT	0.4150	(4)	0.2184	(10)	0.3908	(7)	0.2423	(9)
LT	0.4091	(5)	0.3451	(3)	0.4298	(3)	0.2847	(5)
EL	0.4036	(6)	0.2688	(5)	0.4513	(1)	0.3837	(1)
UK	0.4007	(7)	0.2407	(8)	0.3926	(6)	0.2385	(11)
IE	0.3992	(8)	0.2211	(9)	0.4035	(5)	0.2471	(7)
IT	0.3975	(9)	0.2531	(7)	0.4067	(4)	0.3072	(2)
BE	0.3756	(10)	0.2004	(13)	0.3635	(16)	0.2086	(12)
DE	0.3744	(11)	0.2064	(11)	0.3655	(15)	0.1887	(18)
EE	0.3741	(12)	0.1989	(15)	0.3697	(12)	0.1963	(16)
ES	0.3684	(13)	0.2042	(14)	0.4328	(2)	0.3052	(3)
NO	0.3594	(14)	0.1622	(24)	0.3300	(21)	0.1777	(22)
NL	0.3557	(15)	0.1720	(22)	0.3512	(18)	0.1755	(23)
SK	0.3547	(16)	0.2684	(6)	0.3158	(25)	0.1920	(17)
LU	0.3545	(17)	0.1760	(19)	0.3715	(11)	0.2071	(13)
FR	0.3540	(18)	0.1854	(17)	0.3518	(17)	0.2059	(14)
AT	0.3407	(19)	0.1792	(18)	0.3445	(20)	0.1872	(19)
FI	0.3381	(20)	0.1632	(23)	0.3482	(19)	0.1731	(24)
CZ	0.3379	(21)	0.1877	(16)	0.3222	(24)	0.1384	(25)
DK	0.3325	(22)	0.1720	(21)	0.3676	(13)	0.1969	(15)
SI	0.3291	(23)	0.1603	(25)	0.3279	(22)	0.1811	(21)
CY	0.3262	(24)	0.1990	(14)	0.3844	(9)	0.2685	(6)
SE	0.3132	(25)	0.1746	(20)	0.3278	(23)	0.1844	(20)
IS	0.2968	(26)	0.1263	(26)	0.2898	(26)	0.1348	(26)

Source: Eurostat country code used in EU-SILC (2006) and EU-SILC (2016)

The countries in Table 1 are ranked in descending order according to LB for 2006. That is, those countries in which people are more 'left behind' in terms of multidimensional poverty are ranked first. The greater and closer LB is to 1, the more pressing is the problem of leaving people behind. This problem was especially high in Poland, Latvia and Hungary in 2006 and in Greece, Spain and Lithuania in 2016. Moreover, the AROPE rate is also high in these countries. Hence, the crisp and fuzzy approach lead us to a similar conclusion in these countries and years. In particular, we observe that the LB ranking is similar to the AROPE ranking for 2006 and for 2016 with positive and significant Spearman rank correlation statistics equal to 0.8338 and 0.8824, correspondingly.

Nevertheless, we should not forget that while AROPE informs about the extension of monetary poverty, LB incorporates information on the depth of multidimensional poverty and allows us to evaluate the 'leaving no one behind' notion in terms of multidimensional poverty. In this sense, we also find, for instance, that the AROPE ranking is better than LB ranking for some countries. This is the case of Norway, the Netherlands and Portugal in 2006, or Finland, the Netherlands and the United Kingdom for 2016, which are among the four lowest values of AROPE but move at least 7 positions in the LB ranking. In these countries there are people who are falling behind, even though the countries are above the thresholds of multidimensional poverty, thus highlighting that there is a significant problem of socioeconomic inequality. Likewise, there are other countries where even though the AROPE rate is high, the problem of leaving people behind is not as pressing as in other countries with lower AROPE rates. This is the case of Cyprus and Slovakia in 2006, and Hungary and Slovakia in 2016.

Table 2 reports the changes in LB and the AROPE rate between 2006 and 2016. Poland shows the greatest and significant reduction in LB and AROPE, while the highest increases are found for Spain, Cyprus and Greece. In these countries the change in the extension and in the degree to which people are 'left behind' in terms of multidimensional poverty goes in the same direction. Nonetheless, we can identify countries, such as Portugal and Norway, where the extension of multidimensional poverty (AROPE) increased while the degree to which people are 'left behind' in multidimensional poverty lower. The case of Lithuania is more worrying. If we analyze only the AROPE rate, we conclude that there was a reduction of 6 percentage points from 2006 and to 2016, but this reduction did not lead to an improvement for all, as the average level at which people were 'left behind' increased.

Table 2. Changes in LB and AROPE rate between 2006 and 2016 by

country

Country	$ChangeLB$		$ChangeAROPE$	
PL	−0.0636	*	−0.1631	*
LV	−0.0365	*	−0.1388	*
HU	−0.0535	*	−0.0411	*
PT	−0.0242	*	0.0239	*
LT	0.0208	*	−0.0604	*
EL	0.0477	*	0.1149	*
UK	−0.0080	*	−0.0022	*
IE	0.0043		0.0259	
IT	0.0092	*	0.0541	*
BE	−0.0121	*	0.0082	
DE	−0.0089	*	−0.0177	*
EE	−0.0044	*	−0.0026	
ES	0.0644	*	0.1010	*
NO	−0.0294	*	0.0155	
NL	−0.0045	*	0.0035	*
SK	−0.0389	*	−0.0764	*
LU	0.0170	*	0.0311	*
FR	−0.0022		0.0204	
AT	0.0039		0.0081	
FI	0.0100		0.0099	*
CZ	−0.0157	*	−0.0493	*
DK	0.0351	*	0.0249	*
SI	−0.0012		0.0208	*
CY	0.0582	*	0.0695	*
SE	0.0146	*	0.0098	*
IS	−0.0071		0.0085	

Note: countries ranked in descending order according to LB 2006.
Statistically significant changes are denoted by *

From Table 2, as Figure 1 describes, we get a negative and significant Spearman rank correlation of −0.3901 between LB in 2006 and the change in LB between 2006 and 2016. That is, those countries that reduced LB most are those with the highest values in 2006, so we can talk of convergence across countries.

In this regard, if we had the same individuals in a panel for these 10 years, we could also estimate if there is convergence in the individual LB within each country, apart from the convergence in the average 'left behind' degree across countries.

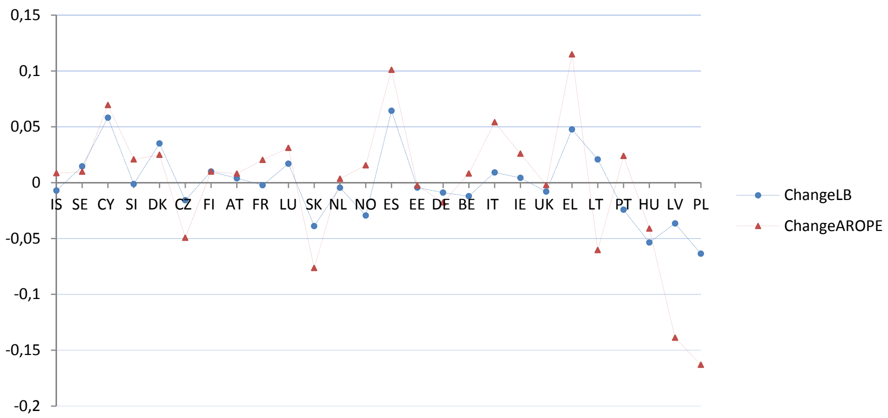

Figure 1: Changes in LB and changes in AROPE for countries in increasing order according to LB for 2006.

Our analysis confirms the need to complement the AROPE rate with a measure that reflects the depth of multidimensional poverty. Our proposal may be an appropriate candidate given its correspondence with the AROPE rate, the ease in interpreting the results and the abundant possibilities it offers for analysis.

5. Conclusions

In this paper we used a fuzzy sets approach to measure of the degree to which an individual is 'left behind' in a specific dimension of poverty and propose alternatives to determine the extent to which an individual is 'left behind' in a multidimensional setting.

This is not the first paper to implement fuzzy sets for measuring multidimensional poverty. However, its novelty lies in proving how fuzzy sets theory can be useful to respond to the concern that progress should not leave anyone behind. Our approach also complements and extends the information on the proportion of individuals who are at the bottom part of the distribution, thus indicating who has been 'left behind' and what degree.

We illustrate our proposal using the AROPE framework on multidimensional poverty. The empirical results highlight that, even though the ranking of countries considering the average 'left behind' level across European countries is in line with that of the AROPE rate our fuzzy measure provides richer information. While AROPE informs about the extension of multidimensional poverty, LB incorporates information on the depth of multidimensional poverty, allowing us to evaluate the 'leaving no one behind' notion in terms of multidimensional poverty. In this regard, some countries have a better (lower) AROPE rate ranking than the LB and viceversa. Our results also reveal some signs of convergence across European countries between 2006 and 2016 in the application of the 'leaving no one behind' principle, as countries with higher LB are those that have reduced their value the most.

The results of our analysis constitute a good starting point for certain socio-economic policies in order to put into practice the 'leaving no one behind' ideal. Obviously more research is needed, for instance, in terms of specific characteristic of the population left behind in each country to properly implement effective policies.

Some caveats should be noted in our proposal. First, we focus on shortfalls that can be small to achieve the objective (those better positioned than the analyzed individual) in each dimension. We could have considered another objective (perhaps the median position or the 90th percentile), but it would have involved the introduction of some value judgment with respect to the objective. Secondly, the choice of one dimension instead of another can lead to different results about the situation of a country or individual. Finally, in the empirical illustration we have taken the union criterion for the joint measurement of dimensions, but, as mentioned in the methodology section, there are other alternatives for aggregating the dimensions, even allowing for different weights for each one. Obviously, each alternative would lead to slightly different results.

All in all, it is clear that, if we want to move towards the achievement of the SDGs and shared prosperity among all, we need to identify those who are left behind, quantify how much they are left behind and act accordingly. Our proposal provides novel insights to progress towards this goal.

6. Appendix

Table $A1$. Number of observations by country and year.

Country	2006	2016
AT	4,004	3,661
BE	4,013	3,689
CY	2,355	2,275
CZ	4,624	4,421
DE	8,585	7,147
DK	3,934	3,487
EE	3,740	3,625
EL	3,148	8,924
ES	7,443	8,260
FI	8,190	6,951
FR	6,766	6,776
HU	4,564	3,898
IE	3,354	2,951
IS	2,222	2,000
IT	12,267	11,272
LT	2,915	2,607
LU	2,879	2,698
LV	2,445	2,786
NL	6,661	7,794
NO	4,486	4,788
PL	10,193	6,834
PT	2,423	5,677
SE	4,912	3,509
SI	6,756	5,643
SK	3,313	3,144
UK	6,131	5,308
	132,323	130,125

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